THE BENDING TO STRETCHING TRANSITION OF A PRESSURIZED BLISTER TEST

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Abstract. The mechanical behavior of a thin blistering film under a uniform pressure changes from a bending plate to a stretching membrane as the thickness and flexural rigidity decrease. An analytical solution is found for the elastic response and the strain energy release rate based on an average membrane stress approximation.

1. Introduction. Cotterell and Chen (1997) derived the strain energy release rate, $G$, for a circular blister under a uniform pressure, $p$, by expanding the membrane stress $N$ as a polynomial of the radial distance $r$ and solving numerically the resulted non-linear differential equation. The dimensionless fracture parameter $\Phi = G/\rho w_0$ was obtained as a function of $W_0 = w_0/h$, where $w_0$ is the blister height and $h$ the film thickness. In this letter, we attempt to compute analytically both the mechanical response and the delamination mechanics by assuming an average membrane stress.

2. Theory. Plate theory by Timoshenko (1959) requires the blister geometry $w(r)$ to be governed by

$$\frac{d^3w}{dr^3} + \frac{1}{r} \frac{d^2w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} - \frac{N}{D} \frac{dw}{dr} = \frac{pr}{2D}$$ (1)

where $D = Eh^3/12 (1 - \nu^2)$, $E$ and $\nu$ are the elastic modulus and the Poisson ratio of the film respectively, and $N = N_e = N_l$ is the average membrane stress independent of $r$. A set of dimensionless parameters are conveniently defined to be $\xi = r/a$, $W = w/h$, $A = \pi a^2$, $\Theta = (a/h) (dw/dr)$, $P = 6(1 - \nu^2) pa^4/Eh^3$, and $k = (Na^2/D)^{1/2}$. Note that bending is predominant at small $k$ and stretching prevails as $k$ increases. Equation (1) is a modified Bessel equation, which can be solved exactly to yield a blister profile of

$$W = \frac{P}{2} \left\{ \frac{2}{k^2} \left[ I_0(k\xi) - I_0(k) \right] \right\} + \frac{1 - \xi^2}{k^2}$$ (2)
Fig. 1. Schematic of a pressurized blister test.

and a blister height of

\[ W_0 = \frac{P}{2} \left[ \frac{2}{k^2 I_1(k)} \left[ \frac{1-I_0(k)}{k^2} + \frac{1}{2} \right] \right] \]  

(3)

with \( I_m(x) \) the \( m^{th} \) order modified Bessel function. It can be shown that the blister profile approaches a spherical cap in the limit \( k \to \infty \). The blister volume \( V \) can be found by integrating (2) with respect to \( r \), where \( V = \int 2\pi r w(r) \, dr \) or

\[ \vartheta = \frac{V}{Ah} = \frac{P}{k^2} \left[ \frac{2}{k^2} + \frac{1}{4} - \frac{I_0(k)}{k I_1(k)} \right]. \]  

(4)

The average membrane stress \( N \) can be computed by putting

\[ N = \frac{Eh}{2a^2(1-v^2)} \int_0^a (dw/dr)^2 \, rdr. \]  

(5)

Substituting (2) into (5) and rearranging, the normalized pressure becomes

\[ P = \frac{k^3}{[6f(k)]^{1/2}} \]  

(6)

with

\[ f(k) = -\frac{(I_0(k)+I_2(k))^2}{8I_1(k)^2} - \frac{2I_2(k)}{kI_1(k)} + \frac{1}{2k^2} + \frac{3}{4}. \]  

(7)

Figure 2 shows \( P(W_0) \). When bending is dominant at small \( k \) and \( W_0 \), \( P(W_0) \) is linear and the gradient \( n = \frac{d(\log P)}{d(\log W_0)} = 1 \). At the other extreme when stretching is dominant at large \( k \) and \( W_0 \), a cubic relation is found with \( n = 3 \). Also shown in figure 2 are the two asymptotes according to the classical solutions for