Estimating Probabilities Relevant to Calculating Relative Risk-Corrected Returns of Alternative Portfolios

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Abstract
In making all-or-none choices between alternative securities, Samuelson (1997b) suggested that investors of different risk-aversion should calculate from past samples of those securities their relevant Harmonic Means, or Geometric means, or other associative means representative of their respective degrees of relative-risk-aversion. Here it is shown how this learning procedure can be improved upon when you have prior knowledge that the securities have log-Normal distributions. Classical estimation theory, concerning consistent, efficient, and sufficient statistics, is shown to have a cash value by means of the calculable measure of (ex ante) “risk-corrected certainty equivalents.” Needed qualifications and testings are also presented.

Key words: Certainty equivalents, learning algorithms

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Samuelson (1997b) nominated a procedure for measuring comparative risk-corrected total returns for alternative investment options, which purports to be superior to conventional Sharpe, Treynor, Modigliani-Modigliani procedures that are based on mean-variance approximations—superior, that is, for investors who approximate to constant-relative-risk-aversion: $-W U'(W)/U(W) =$ constant $> 0$. As a quasi-realistic example—applicable to investors whose $-W U'(W)/U(W) =$ +2—some years of data on A and B, respectively (a) the indexed component of a large pension organization and (b) its actively-managed component, had their respective Harmonic Means calculated: if forced to choose solely A or solely B, that chosen is to be the one with the larger H.M. in the historical sample period.

Friendly questions have been put to me by Duke’s Edwin Burmeister and by MIT’s Robert Solow. Burmeister asked: “If Canadian penny mining stocks had the same H.M. as the S&P 500 Index, would you really be indifferent between them?” In a related vein, Solow asked: “If option B’s total return has an H.M. of 1.05 that involves no variability at all within the sample period, while A has an H.M. of 1.05 that involves much variability within the sample, mightn’t you as a risk-averter find yourself drawn toward favoring B?”

Burmeister of course is hinting at doubts about the stationarity of the probability process generating penny stock price movements. Solow is wondering whether the Harmonic Mean by itself can encompass all that one feels about one’s personal risk tolerance.
(The same query could arise for a Bernoulli with logarithmic utility of wealth: is the Geometric Mean then enough?)

One immediate reaction to Solow runs as follows: (a) After I commit myself to

Maximization of Expected Utility

\[
E[U(W \text{ outcomes})] = E[-1/W],
\]

I will be second guessing my own true reaction to riskiness if I go beyond measuring putative Harmonic Means. (b) In the rare cases of exact ties between H.M._A and H.M._B, it will be harmless to embrace a lexicographic ordering that breaks ties by selecting the option with clearly “less volatility” (somehow objectively measurable). (c) But when H.M._A is even a trifle bigger than H.M._B, my Max \( E[-1/W] \) will not let me second guess that fact, thus implying that realistically I would not be willing to pay anything appreciable for the fetish of breaking ties by using a supplementary criterion of “\textit{ceteris paribus}, eschew gratuitous volatility.”

Do these banalities dispose of the debate, dispose of certain tentative heuristic doubts? On reflection, one wants to go deeper into the problem of how a thoughtful investor forms judgements—not on his/her utility reaction to outcomes—but on his/her judgments about relevant probabilities. This takes one back to old issues of statistical estimation, and maybe to more modern issues of Bayesian prior and posterior personal probabilities of belief.

1. Specifying a possible scenario

In a textbook example, one can first artificially stipulate that \( A \) and \( B \) are “known” to have respectively specified universe probability distributions, \( \{\text{Prob}\{Z_A \leq z\}, \text{Prob}\{Z_B \leq z\}\} = \{P_A(z), P_B(z)\} \), where each \( Z \) is the random variable of that security’s \( 1 + \) total return. Then, if you have thought through and committed yourself to some plausible “sure-thing” Axioms of Ramsey-von Neumann-Marschak-Savage type, you will have already ruled out of court as irrelevant the Burmeister-Solow queries; and you will indeed use the Harmonic Mean when \( U(W) = -1/W \), or the Geometric Mean when \( U(W) = \log W \), or the Arithmetic Mean when you have \textit{linear} risk-neutral utility. There is here no need to have to calculate these means from finite past samples of \( A \) and \( B \) data, since by hypotheses you do “know” the \( P_A(\cdot) \) and \( P_B(\cdot) \) probability distributions and can use \textit{them} to calculate exact universe integrals:

\[
H.M._A = \left[ \int_0^\infty z^{-1}dP_A(z) \right]^{-1}, \text{ etc.}
\]