BIANCHI COSMOLOGIES AS DYNAMICAL SYSTEMS

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Abstract. We discuss specific properties of dynamical systems originating from cosmology and relativity. In particular, we present results of our study of the Bianchi class A cosmological models. We introduce new variables in which the Hamiltonian constraint for all the class A models is solved algebraically. We present results of dimension reduction of the investigated models.

1. Introduction

Models of relativistic cosmology are based on Einstein’s theory of gravitation. The Einstein field equations describe the dynamical evolution of spacetime, as well as the motion of matter and physical fields. They provide a system of coupled, non-linear, partial differential equations. Without some simplifying assumptions or idealization they are intractable by analytical tools. The most natural assumption is to postulate a certain symmetry of space-time. Usually, such idealization allows to reduce Einstein’s field equations to a system of ordinary differential equations. This reduction gives us a possibility to use the rich theory of dynamical systems. It has to be mentioned, however, that dynamical systems of cosmological (or relativistic) origin have many special features which distinguish them from a ‘typical’ dynamical system we meet in dynamical astronomy, classical mechanics or physics. Let us mention a few of them. We can assume that systems we meet have the following form

\[ \dot{x} = v(x), \quad x \in \mathbb{R}^n. \]  

(1)

1. Although the right hand sides of (1) are polynomial, in many cases it is not obvious if the phase flow of this system is complete. It seems that in some cases it is not. It is important to point it out, because for systems with an incomplete flow customary indicators of chaos are not defined, although numerical algorithms do not distinguish between complete and incomplete flows.

2. In many cases the system (1) has a first integral \( H \). Usually, only solutions lying on the level

\[ \mathcal{M} = \{ x \in \mathbb{R}^n \mid H(x) = 0 \}, \]

have a physical interpretation. Thus we have to restrict our system to an invariant set \( \mathcal{M} \). However, in many investigations this point is simply ignored. For example, when we ask for the existence of one or more additional first
integrals then a negative answer for the non-restricted system is not valid for the system restricted to $\mathcal{M}$.

3. In most cases the equilibria of (1) are degenerated and lie on an one or higher dimensional manifold of equilibria.

4. Frequently the system (1) is Hamiltonian with respect to the canonical symplectic form $\omega$ on $\mathbb{R}^n$, $n = 2m$, and the first integral $H$ plays the role of the Hamiltonian function, i.e., $\omega(\cdot, \cdot) = dH$. Usually, the Hamiltonian has the 'natural' form

$$H = \frac{1}{2}g^{ij}p_ip_j + V(q), \quad x = (q^1, \ldots, q^n, p_1, \ldots, p_m) \in \mathbb{R}^{2m},$$

however, the 'kinetic' energy $T = \frac{1}{2}g^{ij}p_ip_j$ is not positive definite. Because of this, there exists no such notion as 'region of possible motion' which plays a fundamental role in studies of natural mechanical systems in classical mechanics. It seems that systems with indefinite kinetic energy possess their own specific properties, however, investigation of these systems is far from being complete.

The last decades gave an immense popularity to the notion of deterministic chaos. Thus, it was natural to look for this phenomenon in the systems mentioned. However, a lot of controversies arose around this subject. On the one hand, these disputes were connected with the numerical character of the obtained results, and, on the other hand, they were caused by some conceptual problems. Here we point out some aspects of this discussion showing our point of view.

The main example of our discussion are class A Bianchi cosmological models. In fact, the controversy about chaotic or non-chaotic behavior concentrates around Bianchi IX model. We describe these models in the next section. In Section 3 we present our original results connected with reduction and simplification of class A Bianchi dynamical systems.

2. Bianchi Class A Cosmologies as Dynamical Systems

Let us assume that space-time has a product topology of type $\mathbb{R} \times \mathcal{M}^3$, where $\mathcal{M}^3$ is 3-dimensional space-like section admitting the action of simply transitive isometry group, i.e. homogeneity group; then Einstein’s equations take the form of a system of ordinary differential equations. The classification of all 3-dimensional homogeneous but anisotropic spaces according to the Lie algebra of the isometry group is called the Bianchi classification (Landau and Lifshitz, 1975).

In this contribution we consider only subclass A of all Bianchi types for which, without any loss of generality, one can assume that the metric of $\mathcal{M}^3$ is diagonal, i.e.

$$ds^2 = (abc)dt^2 - \eta_{ab}(t)(e_i^a dx^i)(e_j^b dx^j),$$

where

$$\eta_{ab} = \text{diag}[a^2(t), b^2(t), c^2(t)], \quad a, b = 1, 2, 3,$$