Success Guarantee of Dual Search in Integer Programming: $p$-th Power Lagrangian Method *

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Abstract. Although the Lagrangian method is a powerful dual search approach in integer programming, it often fails to identify an optimal solution of the primal problem. The $p$-th power Lagrangian method developed in this paper offers a success guarantee for the dual search in generating an optimal solution of the primal integer programming problem in an equivalent setting via two key transformations. One other prominent feature of the $p$-th power Lagrangian method is that the dual search only involves a one-dimensional search within $[0,1]$. Some potential applications of the method as well as the issue of its implementation are discussed.

Key words: Integer programming, Dual search, Lagrangian method

1. Introduction

The following general class of finite integer programming problems is considered in this paper:

\[ \min_{x} f(x) \]  \hspace{1cm} \text{(1.1a)}

\[ \text{s.t. } g_i(x) \leq b_i, \quad i = 1, 2, \ldots, m, \]  \hspace{1cm} \text{(1.1b)}

\[ x \in X \subseteq \mathbb{R}^n. \]  \hspace{1cm} \text{(1.1c)}

where $X$ is a finite integer set. Problem (P) is termed the primal problem. Without loss of generality, $f$ and $g_i$, $i = 1, 2, \ldots, m$, are assumed to be strictly positive for all $x \in X$. Constraints in (1.1b) are called Lagrangian constraints. Define $F$ to be the feasible region of the decision vector $x$ in (P),

\[ F = \{ x \mid g_i(x) \leq b_i, \quad i = 1, 2, \ldots, m; \quad x \in X \}. \]  \hspace{1cm} \text{(1.2)}

Denote by $v(Q)$ the optimal value of an optimization problem (Q). Thus the optimal objective value of the primal problem is $v(P)$.

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The dual search method plays a significant role in integer optimization. The Lagrangian methods are widely used in linear integer programming in finding an optimal solution, see, e.g., Geoffrion (1974), Fisher and Shapiro (1974), Bell and Shapiro (1977), Shapiro (1979), and Fisher (1981). In most situations, the Lagrangian methods provide a lower bound for \( v(P) \). Incorporating the set of Lagrangian constraints into the objective function by introducing a nonnegative Lagrangian multiplier vector, \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}_+^m \), yields a Lagrangian relaxation:

\[
(P_R) \quad \min_{x \in X} L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i [g_i(x) - b_i].
\]

The Lagrangian dual is an optimization problem in \( \lambda \),

\[
(D) \quad \max_{\lambda \in \mathbb{R}^m_+} [v(P_R)].
\]

The Lagrangian method searches for an optimal solution of \((P)\) via maximizing the dual function \(v(P_R))\).

If \( \hat{x} \) solves both \((P)\) and \((P_R)\) with \( \hat{\lambda} \in \mathbb{R}_+^m \), then \( \hat{\lambda} \) is said to be an optimal generating Lagrangian multiplier vector. If \( \hat{x} \) solves both \((P)\) and \((P_R)\) with \( \hat{\lambda} \in \mathbb{R}_+^m \), and \( \hat{\lambda} \) solves the dual problem \((D)\), then \( \{\hat{x}, \hat{\lambda}\} \) is said to be an optimal primal-dual pair of \((P)\).

While the Lagrangian method is a powerful constructive dual search method, it often fails to identify an optimal solution of the primal integer optimization problem. Two critical situations could be present that prevent the Lagrangian method from succeeding in the dual search. Firstly, the optimal solution of \((P)\) may not even be generated by solving \((P_R)\) for any \( \lambda \geq 0 \). Secondly, the optimal solution to \((P_R)\), with \( \lambda^* \) being a solution to the dual problem \((D)\), is not necessarily an optimal solution to \((P)\), or even not feasible. The first situation mentioned above is associated with the existence of an optimal generating Lagrangian multiplier vector. The second situation is related to the existence of an optimal primal-dual pair.

As an illustrative example, let us consider Example 5.12 in Parker and Rardin (1988):

\[
\min \quad 3x_1 + 2x_2
\]
\[
\text{s.t.} \quad g_1(x) = 10 - 5x_1 - 2x_2 \leq 7, \\
 g_2(x) = 15 - 2x_1 - 5x_2 \leq 12, \\
 x \in X = \left\{ x \in \mathbb{Z}^2 : 0 \leq x_1 \leq 2, 0 \leq x_2 \leq 2, 8x_1 + 8x_2 \geq 1 \right\}.
\]

Note that in order to conform with the problem assumption in (1.1) the two Lagrangian constraints in (1.5) take forms equivalent to the original Lagrangian constraints in Ex. 5.12 of Parker and Rardin (1988). The explicit expression of set \( X \) is