Lagrangean Relaxation Revisited, Technical Note

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Abstract. In this paper we consider some theoretical aspects of Lagrangean relaxation which do not seem to appear explicitly in the literature, and add a little more explicit completeness to this area. These concern three key constructs, \( v(IP), v(IP'), v(DP) \) and associated optimality, duality and feasibility aspects of Lagrangean relaxation.

1. Introduction

In this paper we will consider the following class of primal problems.

\[
\text{max} [f(x)] = v(IP) \tag{1.1}
\]

subject to

\[
g(x) \leq 0, \quad x \in X \cap Z^+_n, \tag{1.2}
\]

where

\[
f(\cdot): \mathbb{R}^n \to \mathbb{R}, \tag{1.4}
\]

\[
g(\cdot): \mathbb{R}^n \to \mathbb{R}^l, \tag{1.5}
\]

\[
X \subseteq \mathbb{R}^n,
\]

and we assume that \( f(\cdot), g(\cdot) \) are continuous, and \( X \) compact in some topology.

We define \( S \) to be the set of feasible solutions to problem \( IP \) and we assume that \( S \neq \emptyset \).

We define the following Lagrangean function.

\[
L(\cdot, \cdot): \mathbb{R}^n \times R^l_+ \to \mathbb{R}
\]

\[
L(x, \lambda) = f(x) - \lambda g(x), \quad (x, \lambda) \in \mathbb{R}^n \times R^l_+. \tag{1.6}
\]

The Lagrangean function \( L(\cdot, \cdot) \) defines, for each \( \lambda \in R^l_+ \), a function \( L(\cdot, \lambda) \) on \( \mathbb{R}^n \). The dual vector, \( \lambda \), in effect gives a non-negative weighting factor, \( \lambda_i \), to each constraint \( g_i(x) \leq 0 \) implicit in (1.2), \( i = 1, 2, \ldots, n \), and converts the original objective function \( f(\cdot) \) into a weighted objective function. The purpose of Lagrangean relaxation is to try to find a dual vector value \( \lambda \) for which solving a modified, weighted objective function, problem will lead to a solution of the original primal problem. This is done by creating a dual problem.
The dual function $L(\cdot): R^+_+ \rightarrow R$ is defined as follows

$$L(\lambda) = \max_{x \in X \cap Z^n_+} [L(x, \lambda)], \lambda \in R^+_+,$$

and the optimal dual value $v(DP)$ is given by the dual problem.

$$v(DP) = \min_{\lambda \in R^+_+} [L(\lambda)].$$

In (1.7), (1.8), we use “max” and “min”, rather than “sup” and “inf”, as a result of our assumptions about \{f(\cdot), g(\cdot), X\}.

The continuous relaxation of problem $IP$ is problem $\overline{IP}$ given as follows.

$$\max [f(x)] = v(\overline{IP})$$
subject to

$$g(x) \leq 0,$$

$$x \in X \cap R^n_+.$$ (1.11)

We will define the Lagrangean set valued functions $E(\cdot): R^+_+ \rightarrow Z^n_+, F(\cdot): R^+_+ \rightarrow Z^n_+$, and $F \in Z^n_+$ as follows

$$E(\lambda) = \arg \max_{x \in X \cap Z^n_+} [L(x, \lambda)], \lambda \in R^+_+,$$

$$F(\lambda) = \{x \in E(\lambda): g(x) \leq 0\},$$

$$F = \bigcup_{\lambda \in R^+_+} F(\lambda).$$ (1.14)

We will denote a typical member of $E(\lambda)$ by $x(\lambda)$.

Finally we define the continuous relaxed dual function $\overline{L}(\cdot): \lambda \in R^+_+ \rightarrow R$ by

$$\overline{L}(\lambda) = \max_{x \in X \cap R^+_+} [L(x, \lambda)], \lambda \in R^+_+.$$ (1.15)

$L(\cdot)$ is said to have the integrality [5] property if

$$L(\lambda) = \overline{L}(\lambda), \forall \lambda \in R^+_+.$$ (1.16)

There are well known results concerning $\{v(IP), v(\overline{IP}), L(\cdot), \overline{L}(\cdot)\}$, some of which we will address in this paper, and most of which appear to be addressed to the linear specification of problem $IP$, viz.