The Ground State Energy of a Dilute Two-Dimensional Bose Gas

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The ground state energy per particle of a dilute, homogeneous, two-dimensional Bose gas, in the thermodynamic limit is shown rigorously to be $E_0/N = (\frac{2}{\hbar^2} - 2) |\ln(\rho_0a)|^{-1} + O(|\ln(\rho_0a)|^{-5})$. Here $N$ is the number of particles, $\rho = N/V$ is the particle density and $a$ is the scattering length of the two-body potential. We assume that the two-body potential is short range and nonnegative. The amusing feature of this result is that, in contrast to the three-dimensional case, the energy, $E_0$ is not simply $N(N-1)/2$ times the energy of two particles in a large box of volume (area, really) $V$. It is much larger.

KEY WORDS: Bose gas; two-dimensions; low density; scattering length; ground state energy.

Dedicated to the memory of J. M. Luttinger

1. INTRODUCTION

An ancient problem, going back to the 1950's, is the calculation of the ground state energy of a dilute Bose gas in the thermodynamic limit. The particles are assumed to interact only with a two-body potential and are enclosed in a box of side length $L$. A formula was derived for the energy $E_0(N, L)$ in three dimensions for a two-body potential $v$ with scattering length $a$ (see Appendix) and fixed particle density $\rho = N/V$, ($N =$ particle

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number and $V = \text{volume} = L^3$ in three dimensions). In the thermodynamic limit, the energy/particle is

$$e(\rho) \equiv \lim_{N \to \infty} \frac{E_d(N, \rho^{-1/3} N^{1/3})}{N} \approx 4\pi \mu a$$  \hspace{1cm} (1.1)$$
to lowest order in $\rho$. Here, $\mu = \hbar^2/2m$ with $m$ the mass of a particle.

Our goal here is to derive the analogous low density formula for a two-dimensional Bose gas.

There were several approaches in the 50’s and 60’s to the derivation of the three-dimensional formula (1.1), but none of them were rigorous. Recently we were able to give a rigorous derivation of (1.1) and we refer the reader to ref. 1 for a physically motivated discussion of the essential difficulty in proving (1.1), which, basically, is the fact that at low density the mean interparticle spacing is much smaller than the mean de Broglie wavelength of the particles. Thus, Bose particles cannot be thought of as localized. Furthermore, in ref. 1, we explain rather carefully why the usual expression “perturbation theory” is not appropriate for (1.1)—especially in the hard core case. Indeed, Bogolubov’s 1947 “perturbation theory” yields an estimate, which is incorrect for the low density limit:

$$e(\rho) \approx \frac{1}{2} \rho \int_{\mathbb{R}^3} v$$  \hspace{1cm} (1.2)$$

It was only with a leap of faith that Bogolubov and Landau recognized that $\int v$ is the first Born approximation to $8\pi \mu a$ and thus were able to derive (1.1). Obviously this cannot be called perturbation theory. Moreover, depending on the nature of $v$, it is sometimes the potential energy and sometimes the kinetic energy that is the dominating quantity; for example, in the hard core case the kinetic energy is the perturbation, rather than the potential energy, as the Bogolubov method assumes.

The two-dimensional theory, in contrast, began to receive attention only much later. The first derivation of the correct asymptotic formula was, to our knowledge, done by Schick\textsuperscript{(3)} for a gas of hard discs:

$$e(\rho) \approx 4\pi \mu \rho |\ln(\rho a^2)|^{-1}$$  \hspace{1cm} (1.3)$$

This was accomplished by an infinite summation of “perturbation series” diagrams. Subsequently, a corrected modification of ref. 3 was given in ref. 4. Positive temperature extensions were given in refs. 5 and 6. All this work involved an analysis in momentum space—as was the case for (1.1), with the exception of a method due to one of us that works directly in configuration space.\textsuperscript{(7)} Ovchinnikov\textsuperscript{(8)} derived (1.3) by using, basically, the method in ref. 7. Again, these derivations require several unproven assumptions and are not rigorous.