Inception of Channelization Over a Non-flat Bed

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1. Introduction

The upstream-driven theory for inception of channelization developed by Izumi and Parker [1, 2] for a flow on a flat bed is here extended to the case of a non flat bed. Following the Izumi and Parker theory, the depth-averaged St. Venant equations for shallow water are integrated using a linear perturbative analysis in order to obtain the variation of the finite wavelength corresponding to the distance separation between incipient basins on bed curvature. The threshold concept for bed erosion [3] is used with the aim of associating the excedance of the critical shear stress to the channel head locations. The analysis shows that even for small values of the bed curvature there is a great change in the value of the characteristic wavelength with respect to the flat bed. In particular, an upward concave bed causes an increase of the characteristic wavelength that seems to better match real values [4, 5].

2. Mathematical Model

The flow on a tilted plateau is given by the depth-averaged St. Venant steady equations of shallow water [1, 2]. These equations, written in terms of streamwise momentum, transverse momentum and mass balances, are respectively

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + g \frac{\partial h}{\partial y} = -g \frac{\partial h}{\partial y} - \frac{\tau_{by}}{\rho h} + \frac{\partial}{\partial x} \left( v^\prime \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left( v^\prime \frac{\partial v}{\partial y} \right)
\]

(1a)

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + g \frac{\partial h}{\partial x} = -g \frac{\partial h}{\partial x} - \frac{\tau_{bx}}{\rho h} + \frac{\partial}{\partial x} \left( v^\prime \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( v^\prime \frac{\partial u}{\partial y} \right)
\]

(1b)

\[
\frac{\partial u h}{\partial x} + \frac{\partial v h}{\partial y} = q,
\]

(1c)

where \( u \) and \( v \) are the transverse and streamwise flow velocity (along the \( x \) and \( y \) directions, respectively), \( h \) is the flow depth, \( q \) is the volume rate per unit time per unit bed area, \((\tau_{bx}, \tau_{by})\)
is the vector of the boundary shear stress, \( v_t \) denotes the eddy viscosity associated to shear in the plane of flow, \( g \) is the acceleration due to gravity and \( \rho \) is the water density. Following Izumi and Parker [1] a fully turbulent flow has here been assumed so that the boundary shear stress becomes \((\tau_{bx}, \tau_{by}) = (\rho c_i (u^2 + v^2)^{1/2} u, \rho c_i (u^2 + v^2)^{1/2} v)\), with \( c_i \) friction coefficient. The eddy viscosity is given as \( v_t = \alpha u* h \) where \( \alpha \) is the momentum exchange and \( u* \) is the shear velocity defined so that \( \rho u*^2 = \tau_b \) and \( \tau_b = (\tau_{bx}^2 + \tau_{by}^2)^{1/2} \). The threshold condition is given on the value of the boundary shear stress [3] so that channelization begins when \( \tau_b \) reaches a threshold value \( \tau_{th} \).

Izumi and Parker [1] proposed a method for the integration of (1a–c) over a transverse perturbation of a flat bed with constant slope. The same theory is applied in this note over a non-flat bed. This is obtained through a series expansion of the bed so that

\[
\eta = \eta_d - a \cos kx - sy - s_1 y^2 + O(y^3),
\]

where \( \eta(x, y) \) is the perturbed bed, \( \eta_d \) denotes the constant unperturbed elevation of the divide, \( a \) and \( k = 2\pi/\lambda \) are, respectively, the infinitesimal amplitude and the wavenumber of the surface perturbation, \( s \) is the unperturbed slope and \( s_1 \) is related to the curvature of the bed: \( s_1 > 0 \) and \( s_1 < 0 \) mean a downward or an upward concave bed, respectively. The values for \( s \) and \( s_1 \) should be relatively small so that the hypothesis of quasi-steady flow is still physically reasonable; moreover, a relative small value for \( s_1 \) gives a not so high local bed slope and curvature so that the infinitesimal-slope formulation is satisfied. In such a way one can neglect the geometric nonlinearities related to high bed slopes in the modeling, while, at the same time, important dynamic nonlinearities are retained. It is hypothesized that the curvature is pre-existent to the here investigated process (for example due to geological causes).

The normalization of the equations is done according to the characteristic scales of the problem. In particular, using the base normal flow hypothesis by setting \( a = 0 \) and dropping all dependency on \( x \) [6] and the backwater terms [1], the distance \( L_{th} \) downstream of the divide at which the base normal flow reaches the threshold condition and the correspondent streamwise flow velocity \( V_{th} \) and flow depth \( H_{th} \) can be found, that is

\[
V_{th} = \sqrt{\frac{\tau_{th}}{\rho c_f}}, \quad L_{th} = \frac{N}{4gqs_1}, \quad H_{th} = \frac{N}{4gqs_1 V_{th}}
\]

with \( N = -gqs + \sqrt{g^2 q^2 s^2 + 8c_i gqs_1 V_{th}^3} \). The Froude number corresponding to the threshold length becomes

\[
F_{th}^2 = \frac{V_{th}^2}{g H_{th}} = \frac{s}{c_f} + \frac{2s_1 L_{th}}{c_f} = F_1^2 + F_2^2.
\]

The following adimensionalizations are then introduced: \((u, v) = V_{th}(\hat{u}, \hat{v}), (h, \eta, a) = H_{th}(\hat{h}, \hat{\eta}, \hat{a}), (x, y) = L_{th}(\hat{x}, \hat{y})\) and \( k = \hat{k}/L_{th}, \) where the hat denotes a nondimensional quantity. With these new variables and remembering the relationship (2), the equations (1a–c) become

\[
(F_1^2 + F_2^2) \left( k^* u \frac{\partial v}{\partial \phi} + \psi v \frac{\partial v}{\partial y} \right) + \psi \frac{\partial h}{\partial y} + \psi \frac{\partial h}{\partial y} + \psi \frac{\partial h}{\partial y} + 
\]

\[
+ \epsilon \frac{F_1^2 + F_2^2}{F_1^2} \left[ k^* \frac{\partial}{\partial \phi} \left( \chi \frac{\partial v}{\partial \phi} \right) + \psi^2 \frac{\partial}{\partial y} \left( \chi \frac{\partial v}{\partial y} \right) \right],
\]

(5a)