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An Elementary Approach to Exponential Spaces

Dedicated to Horst Herrlich on the occasion of his 60th birthday

EVA LOWEN-COLEBUNDERS
Departement Wiskunde, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussel, Belgium

GÜNTHER RICHTER
Fakultät für Mathematik, Universität Bielefeld, Universitätsstrasse 25, D-33615 Bielefeld, Germany

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Abstract. In 1970, Day and Kelly characterized exponential spaces by a condition (C). Eight years later, Hofmann and Lawson pointed out that this is equivalent to quasi-local compactness, i.e. every neighborhood V of a point contains a smaller one W such that any open cover of V admits a finite subcover of W. These characterizations work with topologies on topologies and may be felt to be not really elementary. This note instead offers an elementary approach which applies to quotient-reflective subcategories as well and includes a natural generalization of the compact-open topology on function spaces.

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0. Introduction

In [4], Day and Kelly characterized those topological spaces X for which the endofunctor $X \times -$ of the category **Top** of topological spaces preserves quotients by a condition (C) of a certain topology on the topology of X. In the sequel, (C) was called *core-compactness* [13] or, equivalently, *quasi local compactness* [18]. The latter means that every neighborhood V of a point contains a smaller one W which is *relatively compact under* V, denoted by $W \ll V$. This abbreviates that any open cover of V admits a finite subcover of W or every filter on W has a cluster point in V, respectively [13, Proposition 4.2]. Although not mentioned in [4], these spaces X are just the *exponential objects* in **Top**, i.e. such that $X \times -$ is left-adjoint [8]. The corresponding right-adjoint $-^X$ yields canonical topologies on function spaces $C(X, Y) = Y^X$.

Even Isbell considered the original proof to be 'rather tricky' and made several attempts at obtaining further insight, see [14], for further references. Especially, an elementary description of the canonical topologies on function spaces was missing. Section 1 offers a natural generalization of the well-known *compact-open topology*

for that. This yields an elementary proof for quasi-locally compact spaces X to be exponential similar to the locally compact case [7]. In addition, the *interpolation property* [8], i.e. for $W \ll V$ there is some U with $W \ll U \ll V$, turns out to be essential. Likewise, there is an elementary proof for $X \times -$ to preserve quotients, despite the fact that this is immediate from the exponentiability [cf. 5, 4.1]. This works for quotient-reflective subcategories as well.

The second section deals with the reverse direction. Again there is an elementary proof for an exponential object in **Top** to be quasi-locally compact, adapting a construction used by Herrlich [11, 7.3.18] for the Hausdorff case. Altogether, the exponential objects in any quotient-reflective subcategory **Y** of **Top** turn out to be the respective quasi-locally compact spaces. The same holds for various initial hulls, as demonstrated in the last section.

The characterization of exponential objects in quotient-reflective subcategories of **Top** was originally due to F. Cagliari [2]. Her proof needs the result of Day and Kelly [4] as well as results from a joint paper with S. Mantovani [3] that deals with pseudotopological spaces and works with a characterization of exponential morphisms by partial products [6]. This approach is far from being elementary in the sense of classical set-theoretical topology.

Note, however, that there is a nice categorical proof for the necessity of quasi-local compactness hidden in the characterization of exponential morphisms p: $X \to T$ in **Top** by S. Niefield [16] as a special case $T = \mathbf{1}$. Unfortunately, this proof depends essentially on the presence of the Sierpinski space $\mathbf{2}$. It is not available in quotient-reflective subcategories like Hausdorff spaces.

All subcategories are assumed to be full and isomorphism-closed. Categorical terminology follows [1, 12].

1. Quasi Locally Compact Spaces are Exponential

The *compact-open topology* on the function space $C(X, Y)$ of all continuous maps from X to Y is given by the subbase of all sets

$$(C, O) := \{f \in C(X, Y) \mid C \subseteq f^{-1}(O)\},$$

$X \supseteq C$ compact, $Y \supseteq O$ open. Replacing '$f^{-1}(O) \supseteq C$ compact' by '$f^{-1}(O) \gg C$' yields the *relatively compact-open topology* on $C(X, Y)$, with subbase sets

$$\langle C, O\rangle := \{f \in C(X, Y) \mid C \ll f^{-1}(O)\}, \quad Y \supseteq O \text{ open}.$$

It refines the topology of pointwise convergence, because $\{x\} \subseteq f^{-1}(O) \Leftrightarrow \{x\} \ll f^{-1}(O)$, hence $C(X, Y)$ is contained in any quotient-reflective subcategory of **Top** that contains Y.

PROPOSITION 1.1. *Let X be locally compact (i.e. every point of X has a neighborhood base of compact sets). Then, for any space Y, the relatively compact-open topology on $C(X, Y)$ coincides with the compact-open topology.*