**L_p-ESTIMATES FOR SOLUTIONS TO THE INITIAL BOUNDARY-VALUE PROBLEM FOR THE GENERALIZED STOKES SYSTEM IN A BOUNDED DOMAIN**

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We consider the initial boundary-value problem

\[ \frac{\partial \vec{v}}{\partial t} + A(x, t, \partial / \partial x) \vec{v} + \nabla p = f(x, t), \]

\[ \nabla \cdot \vec{v} = 0, \quad x \in \Omega \subset \mathbb{R}^n, \quad t \in (0, T), \]

\[ \vec{v}(x, 0) = \vec{v}_0(x), \quad \vec{v}(x, t)|_{x \in S} = \vec{a}(x, t) \]

in a bounded domain \( \Omega \subset \mathbb{R}^n, n \geq 2 \), with boundary \( S = \partial \Omega \subset C^3 \) consisting of \( m \) connected components \( S_k, k = 1, \ldots, m \). If \( A = -\Delta I \), then (1.1), (1.2) is the Stokes system. The main result of the paper is the proof of the solvability of the problem (0.1)–(0.3) in anisotropic Sobolev spaces. Bibliography: 23 titles.

§ 1. Introduction

We consider the initial boundary-value problem

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in a bounded domain \( \Omega \subset \mathbb{R}^n, n \geq 2 \), with boundary \( S = \partial \Omega \subset C^3 \) consisting of \( m \) connected components \( S_k, k = 1, \ldots, m \), so that \( \mathbb{R}^n \setminus \overline{\Omega} = \Omega_1 \cup \ldots \cup \Omega_{m-1} \cup \Omega_m \), where \( \Omega_1, \ldots, \Omega_{m-1} \) are bounded domains with boundaries \( \partial \Omega_k = S_k, k = 1, \ldots, m-1 \), \( \Omega_m \) is an “exterior” domain, and \( \partial \Omega_m = S_m \). The vector field \( \vec{v}(x, t) = (v_1, \ldots, v_n) \) and the function \( p(x, t) \) are unknown. We denote by \( A(x, t, \partial / \partial x) \) a matrix elliptic-type differential operator with real coefficients depending on \( x \) and \( t \). The principle part of the operator containing the second-order derivatives is denoted by \( A_0 \). We assume that for all \( x \in \overline{\Omega}, t \in [0, T], \xi \in \mathbb{R}^n \) the matrix \( A_0(x, t, i\xi) \) is positive-definite, i.e.,

\[ C^{-1}|\xi|^2|\eta|^2 \leq A_0(x, t, i\xi)\eta \cdot \eta \leq C|\xi|^2|\eta|^2 \quad \forall \xi, \eta \in \mathbb{R}^n \]

for some \( C > 0 \) independent of \( \xi \) and \( \eta \). If \( A = -\Delta I \), then the system (1.1), (1.2) is the Stokes system. More general systems appear in the linearization of the equation of motion of non-Newtonian liquids [1, 2].

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The main result of this paper is to prove the solvability of the problem (1.1)–(1.3) in anisotropic Sobolev spaces. We recall the required definitions. Let $G$ be a domain in $\mathbb{R}^n$ with smooth boundary. By $W^k_p(G)$, $p > 1$, where $k$ is an integer, we mean the space of functions having generalized derivatives of order up to $k$: $\mathcal{D}^j u \in L_p(G)$, $0 \leq |j| \leq k$, with the norm
\[
\|u\|_{W^k_p(G)}^p = \sum_{i,j \leq k} \int_G |\mathcal{D}^j u(x)|^p \, dx.
\]

We assume that $W^0_p(G) = L_p(G)$. The principle part of the norm containing only the $k$th order derivatives is denoted by $\|u\|_{W^k_p(G)}$:
\[
\|u\|_{W^k_p(G)} = \left( \sum_{|j|=k} \int_G |\mathcal{D}^j u(x)|^p \, dx \right)^{1/p}.
\]

If $l = [l] + \lambda$, $0 < \lambda < 1$, then $W^l_p(G)$ is the space with the norm
\[
\|u\|_{W^l_p(G)}^p = \|u\|_{W^{[l]}_p(G)}^p + \|u\|_{\tilde{W}^l_p(G)}^p,
\]
where
\[
\|u\|_{\tilde{W}^l_p(G)} = \sum_{|j|=|l|} \int_G \int_{G_j} \frac{|\mathcal{D}^j u(x) - \mathcal{D}^j u(y)|^p \, dx \, dy}{|x-y|^{n+pl}}.
\]

If $l$ is not an integer, then this space coincides with the Besov space $B^l_p(G)$. However, $B^l_p(G) \neq W^l_p(G)$ if $l$ is an integer and $p \neq 2$ (cf. [3]). By $\tilde{W}^l_p(G)$ we mean the space of functions admitting the zero extension to $\mathbb{R}^n \setminus G$ such that the extended function is of the same class, and
\[
\|u\|_{\tilde{W}^l_p(G)} = \|u\|_{W^l_p(\mathbb{R}^n)} \quad (u(x) = 0, \quad x \in \mathbb{R}^n \setminus G).
\]

If $u \in W^l_p(G)$, then $\mathcal{D}^j u|_{\partial G} \in W^{l-j-1/p}_{p}((\partial G)$ for $l-j-1/p > 0$ ($\mathcal{D}^j u|_{\partial G} \in B^{l-j-1/p}_{p}(\partial G)$ if $l-j-1/p$ is an integer). On $\partial G$, as well as on other smooth manifolds, the norms in the Sobolev spaces are defined with the help of local charts and partitions of unity in a standard way.

By an anisotropic space $W^{l/2}_p(G \times (0, T))$ we mean the space of functions $u(t,x)$, $x \in G$, $t \in (0, T)$, with the finite norm
\[
\|u\|_{W^{l/2}_p(G \times (0, T))}^p = \int_0^T \|u(\cdot, t)\|_{W^l_p(G)}^p \, dt + \int_G \|u\|_{W^{l/2}_p(0, T)}^p \, dx.
\]

In other words,
\[
W^{l/2}_p(G \times (0, T)) = L_p(0, T; W^l_p(G)) \cap L_p(G; W^{l/2}_p(0, T)) = L_p(0, T; W^l_p(G)) \cap W^{l/2}_p(0, T; L_p(G)).
\]

If $u \in W^{l/2}_p(G \times (0, T))$, $|j| + 2k \leq l$, then
\[
\mathcal{D}^j_x \mathcal{D}^k_t u \in W^{l-|j|-2k, (1/2)(l-|j|-2k)}_p(G \times (0, T)).
\]