**L_p-ESTIMATES FOR A SOLUTION TO THE NONSTATIONARY STOKES EQUATIONS**

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$L_p$-estimates for the nonstationary Stokes problem are established. Bibliography: 14 titles.

§ 1. Introduction

We consider the initial-boundary-value problem

$$
\frac{\partial \vec{v}}{\partial t} - \Delta \vec{v} + \nabla p = \nabla \cdot \mathbf{F}(x,t), \quad \nabla \cdot \vec{v} = 0, \quad x \in \Omega \subset \mathbb{R}^n, \quad t \in (0,T),
$$

$$
\vec{v}(x,0) = 0, \quad \vec{v}(x,t) |_{x \in S} = \vec{a}(x,t)
$$

(1.1)

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with boundary $S \subset \mathbb{C}^2$, where $\mathbf{F}(x,t)$ is a matrix with entries $F_{ik}(x,t)$, $i,k = 1, \ldots, n$, and $\nabla \cdot \mathbf{F} = \left( \sum_{i=1}^{n} \frac{\partial F_{ik}}{\partial x_i} \right)_{k=1,\ldots,n}$. We assume that $\mathbf{F}$ and $\vec{a}$ satisfy the following conditions:

$$
\mathbf{F} \in L_p(Q_T), \quad Q_T = \Omega \times (0,T), \quad p > 1, \quad \vec{a} \in W^{1-1/p, 1/2-1/(2p)}(\Sigma_T), \quad \Sigma_T = S \times (0,T),
$$

$$
\int_{S} \vec{a} \cdot \vec{n} \, dS = 0, \quad \vec{a} \cdot \vec{n} = \text{div}_S \vec{A}, \quad \vec{A} \in W^{1/2}(0,T; W^{1-1/p}(S)).
$$

(1.2)

By $W_{p}^{l}(\Omega)$, $W_{p}^{l/2}(Q_T)$ and so on we mean the standard Sobolev–Slobodetskii isotropic and anisotropic spaces defined, for example, in [1]. We denote by $W_{p,0}^{l/2}(Q_T)$ and $W_{p,0}^{l/2}(\Sigma_T)$ the sets of elements in $W_{p}^{l/2}(Q_T)$ and $W_{p}^{l/2}(\Sigma_T)$ that admit the zero extension to $\Omega \times (-\infty,T)$ and $\Sigma \times (-\infty,T)$ such that the extended functions are of the same class and the following estimates hold:

$$
\|u\|_{W_{p,0}^{l/2}(Q_T)} = \|u_{0}\|_{W_{p,0}^{l/2}(\Omega \times (-\infty,T))},
$$

$$
\|u\|_{W_{p,0}^{l/2}(\Sigma_T)} = \|u_{0}\|_{W_{p,0}^{l/2}(S \times (-\infty,T))},
$$

where $u_{0}(x,t) = u(x,t)$ for $t > 0$ and $u_{0}(x,t) = 0$ for $t < 0$. If $u \in W_{p}^{l/2}(Q_T)$ or $u \in W_{p,0}^{l/2}(Q_T)$, then $u \in W_{p}^{l-2-r/2}(0,T; W_{p}^{l}(\Omega))$ or $u \in W_{p,0}^{l-2-r/2}(0,T; W_{p}^{l}(\Omega))$, $r < l$, respectively.

We refer the reader to [2] for representations of functions on $S$ in the form $\text{div}_S \vec{A}$.

The main result of the paper is contained in the following assertion.

**Theorem 1.1.** Let $\Omega$ be a bounded convex domain with boundary $S \subset \mathbb{C}^2$. Then the solution to the problem (1.1) satisfies the inequality

$$
\|\vec{v}\|_{W_{p,0}^{l/2}(Q_T)} \leq c(T) \left( \|\mathbf{F}\|_{L_p(Q_T)} + \|\vec{a}\|_{W_{p,0}^{1-1/p, 1/2-1/(2p)}(\Sigma_T)} + \|\vec{A}\|_{W_{p,0}^{1/2}(0,T; W_{p}^{1-1/p}(S))} \right) \equiv c \mathcal{N}.
$$

(1.3)

The pressure \( p(x,t) \) can be represented in the form

\[
p(x,t) = p_1(x,t) + \frac{\partial p(x,t)}{\partial t},
\]

where \( p(x,t) \) is a harmonic function and

\[
\|p_1\|_{L^p(\Omega_T)} + \|\nabla p\|_{W^{1/2}_p(\Omega_T)} \leq c N.
\]

Moreover, if \( \tilde{A} \in W^{1-1/(2p)-r/2}_p(0,T;W^r_p(S)) \), \( r \in (0,1-1/p) \) (in particular, if \( \tilde{a} \cdot \tilde{n} = 0 \) and \( \tilde{A} = 0 \), then

\[
\|P\|_{W^{1-1/(2p)-r/2}_p(0,T;W^r_p(S))} \leq c(T)(N + \|\tilde{A}\|_{W^{1-1/(2p)-r/2}_p(0,T;W^r_p(S))}).
\]

The constants in (1.3)–(1.5) are nondecreasing functions of \( T \).

The assumption that \( \Omega \) is convex is explained by the fact that for the representation of a solution to the problem (1.1) with \( S = 0 \) the hydrodynamical potentials are used [3–5]. In the case of a half-space, the explicit formulas for a solution to this problem allow us to obtain a stronger result, namely, the boundedness of the norm \( \|P\|_{W^{1-1/(2p)-r/2}_p(0,T;W^r_p(S))} \) (in this case, \( S = \mathbb{R}^{d-1} \)). The problem (1.1) with \( \tilde{a} = 0 \) was considered in [6, 7]. In [6], the estimate (1.3) was obtained under the assumption that all the vector fields \( \tilde{F}_n = (F_{ik}, \ldots, F_{nk}) \), \( k = 1, \ldots, n \), are solenoidal and \( \tilde{F}_k \cdot \tilde{n} = 0 \) (in the weak sense). As was shown in [7], in the case \( p = 2 \), the pressure can be defined as the sum \( p(x,t) = p_1(x,t) + p_2(x,t) \), where \( p_1 \in L_p(\Omega_T) \) and \( p_2 \in W^{-1/2}_2(0,T;W^1_2(\Omega)) \).

In Secs. 2–6, we analyze the problem (1.1) with \( S = 0 \). In Sec. 7, we prove Theorem 1.1 and discuss possible generalizations of this theorem, an elementary proof of the representation (1.4) for pressure (perhaps, with some other \( p_1 \) and \( P \)), and the estimate (1.5) (but not (1.6)) for \( p = 2 \).

\section*{§ 2. Problem (1.1) with \( S = 0 \)}

We consider the problem

\[
\begin{align*}
\tilde{v}_t - \Delta \tilde{v} + \nabla p &= 0, \quad \nabla \cdot \tilde{v} = 0, \quad x \in \Omega, \ t \in (0,T), \\
\tilde{v}|_{t=0} &= 0, \quad \tilde{v}|_{x \in S} = \tilde{a}(x,t),
\end{align*}
\]

(2.1)

where \( \tilde{a}(x,t) \) is a smooth function such that

\[
\int_S \tilde{a} \cdot \tilde{n} dS = 0, \quad \tilde{a}(x,0) = 0.
\]

As is known [3–5], in the case of a convex domain \( \Omega \), the solution to the problem (2.1) can be written in the form

\[
\tilde{v}(x,t) = \tilde{U}(x,t) + \nabla V(x,t),
\]

(2.2)