**Abstract.** We prove that Hill’s lunar problem does not possess a second analytic integral of motion, independent of the Hamiltonian. In order to obtain this result, we avoid the usual normalization in which the angular velocity $\omega$ of the rotating reference frame is put equal to unit. We construct an artificial Hamiltonian that includes an arbitrary parameter $b$ and show that this Hamiltonian does not possess an analytic integral of motion for $\omega$ in an open interval around zero. Then, by selecting suitable values of $\omega, b$ and using the invariance of the Hamiltonian under scaling in the units of length and time, we show that the Hamiltonian of Hill’s problem does not possess an integral of motion, analytically continued from the integrable two–body problem in a rotating frame.

**Key words:** Hill’s lunar problem, non-integrability.

1. Introduction

Hill’s lunar problem is characterized by the following assumptions: We consider the three–body problem with masses $m_1, m_2, m_3$ (say: Sun, Earth, Moon) in the hierarchy $m_1 \gg m_2, m_1 \gg m_3$, where the mass ratio $m_i/(m_2 + m_3), i = 2, 3$ is arbitrary. Moreover, we assume that the solar parallax and eccentricity and the lunar inclination vanish, as Hill (1878, p. 13) first described the particular case of the three-body problem named after him. This system is applicable to several real existing situations in celestial mechanics and can be regarded as the probably simplest non-integrable (as we will prove) many body problem.

There are various interesting possibilities to derive the equations of Hill’s lunar problem (Hill, 1878; Spirig and Waldvogel, 1985). We choose here a special direct derivation (Szebehely, 1967, p. 609) of Hill’s problem. That is we consider the restricted three–body problem and let the mass ratio of the two primaries $\mu = m_2/(m_1 + m_2)$ tend to zero, while at the same time the vicinity of the second primary is enlarged by a factor $\mu^{-1/3}$. This problem is described, in polar coordinates, by the Hamiltonian

$$H = \frac{1}{2} \left( p_\rho^2 + \frac{p_\theta^2}{\rho^2} \right) - \frac{1}{\rho} - p_\theta + \frac{\rho^2}{2}(1 - 3 \cos^2 \theta).$$

(1)
In order to introduce an arbitrary parameter in the Hamiltonian of Hill’s problem in a natural way, we rewrite the equations of motion of the circular restricted three-body problem, without performing the usual normalization of units, in which the angular velocity of the rotating frame $\omega$ and the gravitational constant $k$ are put equal to 1. We keep only the unit of mass fixed by the equation $m_1 + m_2 = 1$ and obtain the corresponding Hamiltonian $H_\omega$. In order to remove the degeneracy of the two–body problem in the inertial frame, we construct an artificial Hamiltonian that includes an arbitrary parameter $b$. By a method similar to that developed in Meletlidou and Ichtiaroglou (1994a,b), we prove that this Hamiltonian does not possess an integral of motion, which is analytic in the canonical variables and analytic in $\omega$ in an open interval around $\omega = 0$. Then we fix the value of $\omega$ in the above interval and select a suitable value for $b$. Finally, performing an arbitrary scaling in the units of length and time, we show that the Hamiltonian $H$ does not possess an analytic integral of motion, independent of $H$, that is continued analytically from the two–body problem in a rotating frame of reference.

2. The Hamiltonian of Hill’s Problem

We consider the usual planar circular restricted three–body problem, with the two primaries $P_1$, $P_2$, with masses $1 - \mu$ and $\mu$ respectively, situated on the rotating $x$–axis at the points $x_1 = -\mu R$ and $x_2 = (1 - \mu)R$, where

$$R = \left(\frac{k}{\omega^2}\right)^{1/3}$$

(2)

is the mutual distance of the primaries, $k > 0$ is the gravitational constant and $\omega$ is the angular velocity of the rotating frame. The equations of motion of the third body are

$$\ddot{x} = 2\omega \dot{y} + \omega^2 x - \frac{\partial V}{\partial x},$$

$$\ddot{y} = -2\omega \dot{x} + \omega^2 y - \frac{\partial V}{\partial y},$$

(3)

where

$$V = -\frac{k(1 - \mu)}{r_1} - \frac{\mu k}{r_2}$$

(4)

and

$$r_1 = \sqrt{(x + \mu R)^2 + y^2},$$

$$r_2 = \sqrt{(x - (1 - \mu)R)^2 + y^2}.$$  

(5)