TESTING FOR UNIFORMITY OF THE RESIDUAL LIFE TIME BASED ON DYNAMIC KULLBACK-LEIBLER INFORMATION

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Abstract. In this article a goodness of fit test for distributional assumptions regarding the residual lifetime is proposed. The test is based on a Vasicek type sum log-spacings estimators of a dynamic version of Kullback-Leibler information. The specific distributional hypothesis considered is of the uniformity over [0,1]. However, the test can be used for testing any simple goodness of fit hypothesis. The asymptotic distribution of the test statistic together with a tabulation of the critical points for different sample sizes are given. Finally, the power function of the test is empirically studied in comparison with some competitors, and the test appears to be meritorious.

Key words and phrases: Discrimination information, uniformity tests, goodness-of-fit tests, consistent tests.

1. Introduction

Let $T$ be a random variable representing time to failure of a system. For example, it might be the time to failure of a bio-system or the time to failure of an engineering system. Let $F(t) = P(T \leq t)$ be the lifetime distribution of $T$ with the survival function $\tilde{F}(t) = 1 - F(t)$. We assume that $F$ is differentiable with density function $f(t)$ concentrated on the interval $[0,1]$. Consider the problem of testing the null hypothesis $H_0$ that the residual lifetime distribution of a system given that it has survived until time say $t_0$ is the uniform over $(t_0,1)$. That is, given that the age of a system is $t_0$ we want to test

$$H_0 : \frac{\tilde{F}(x + t_0)}{\tilde{F}(t_0)} = \frac{1 - x - t_0}{1 - t_0} \quad \text{for all} \quad 0 \leq x \leq 1 - t_0.$$  \hspace{1cm} (1.1)

The alternative to $H_0$ is

$$H_a : \frac{\tilde{F}(x + t_0)}{\tilde{F}(t_0)} \neq \frac{1 - x - t_0}{1 - t_0} \quad \text{for at least one} \ x \in \ [0, 1 - t_0].$$  \hspace{1cm} (1.2)

If $H_0$ states that $\frac{\tilde{F}(x + t_0)}{\tilde{F}(t_0)} = \frac{F_0(x + t_0)}{F_0(t_0)}$ for all $x > 0$, where $\tilde{F}_0$ is continuous and fully specified distribution, then our test of uniformity also allows one to test $H_a^*$; see Lemma 1. The main purpose of this paper is to propose a method for testing (1.1) against (1.2).

In Section 2 the test statistic based on the dynamic version of Kullback-Leibler information is formulated and its main properties are stated. An advantage of our test statistic is that it incorporates information about the age and therefore it can be used for testing of a certain probability model for the residual lifetime distribution. In Section 3,
the percentage points of our test statistic were estimated for various sample sizes and levels, discussed how to implement the proposed test and gave an illustrative example. Section 3 also compares the powers of the proposed test with other competing tests. Finally in Section 4, we derive the asymptotic behaviors of our test statistic.

2. Test statistic

To discriminate between the two hypotheses (1.1) and (1.2), we use the dynamic version of Kullback-Leibler discrimination information function between two residual lifetime distributions given by

\[
K(F, F_0; t_0) = \int_{t_0}^{1} \frac{f(x)}{\bar{F}(t_0)} \log \left( \frac{f(x)}{\bar{F}(t_0)} \right) dx
\]

\[
= \log \bar{F}(t_0) + H(F; t_0) - \int_{t_0}^{1} \frac{f(x)}{\bar{F}(t)} \log f_0(x) dx,
\]

where

\[
H(F; t_0) = \int_{t_0}^{1} \frac{f(x)}{\bar{F}(t_0)} \log \frac{f(x)}{\bar{F}(t_0)} dx
\]

\[
= \frac{1}{\bar{F}(t_0)} \left[ \int_{t_0}^{1} f(x) \log f(x) dx \right] - \log \bar{F}(t_0).
\]

It is well known that \( K(F, F_0; t_0) \geq 0 \) and the equality holds if and only if \( \frac{f(x)}{\bar{F}(t_0)} = \frac{f_0(x)}{\bar{F}_0(t_0)} \) for all \( x \geq t_0 \); Ebrahimi (1996) and Ebrahimi and Kirmani (1996a, 1996b).

In our situation, discriminating between \( \frac{\bar{F}(x+t_0)}{\bar{F}(t_0)} \) the true residual survival distribution at age \( t_0 \) and the corresponding \( \frac{\bar{F}_0(x+t_0)}{\bar{F}_0(t_0)} = \frac{1-x-t_0}{1-t_0} \) the residual survival distribution for the uniform \((0,1)\), the equation (2.1) reduces to

\[
K(F, F_0; t_0) = \log (1 - t_0) + H(F; t_0).
\]

The discrimination function (2.3) is a measure of disparity between residual lifetime distribution at age \( t_0 \) and corresponding residual lifetime distribution for the uniform \((0,1)\). Under the null hypothesis \( H_0 \) in (1.1) \( K(F, F_0; t_0) = 0 \), and large values of \( K(F, F_0; t_0) \) favor \( H_a \).

Since evaluation of (2.3) requires complete knowledge of \( F \), then \( K(F, F_0; t_0) \) is not operational. We operationalize \( K(F, F_0; t_0) \) by developing the discrimination information statistic as follows.

Given a random sample \( T_1, \ldots, T_n \) from \( F \), let \( T(1), \ldots, T(n) \) be the \( n \) ordered observations. To estimate \( H(F; t_0) \) in (2.3) we write

\[
H(F; t_0) = -\log \bar{F}(t_0) - \frac{1}{\bar{F}(t_0)} \int_{t_0}^{1} \log \left( \frac{d}{dp} F^{-1}(p) \right) dp.
\]