TWISTS AND SINGULAR VECTORS IN $\hat{\mathfrak{sl}}(2|1)$ REPRESENTATIONS

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We propose new formulas for singular vectors in Verma modules over the affine Lie superalgebra $\hat{\mathfrak{sl}}(2|1)$. We analyze the coexistence of singular vectors of different types and identify the twisted modules $N_{h,k,\theta}^{\pm}$ arising as submodules and quotient modules of $\hat{\mathfrak{sl}}(2|1)$ Verma modules. We show that with the twists (spectral flow transformations) properly taken into account, a resolution of irreducible representations can be constructed consisting of only the $N_{h,k,\theta}^{\pm}$ modules.

1. Introduction

In this work, we consider elements of the representation theory of the affine Lie superalgebra $\hat{\mathfrak{sl}}(2|1)$. The affine $\mathfrak{sl}(2|1)$ symmetry emerges in the models of disordered systems introduced in relation to the integer quantum Hall effect [1]–[3]. The $\hat{\mathfrak{sl}}(2|1)$ algebra is also interesting because, in a certain sense, it combines the characteristic features of the affine Lie algebra $\hat{\mathfrak{sl}}(2)$ and the $\mathcal{N}=2$ superconformal algebra. It is related to the latter via Hamiltonian reduction [4]–[6] and its “inversion” [7]; a relation between the $\hat{\mathfrak{sl}}(2|1)$ and $\hat{\mathfrak{sl}}(2)$ algebras, apart from the obvious subalgebra embedding, was worked out in [8], where $\hat{\mathfrak{sl}}(2|1)$ was shown to be a vertex-operator extension of the sum of two $\hat{\mathfrak{sl}}(2)$ algebras with “dual” levels $k$ and $k'$ such that $(k+1)(k'+1) = 1$. The construction in [8] has led to a decomposition formula for $\hat{\mathfrak{sl}}(2|1)$ representations in terms of $\hat{\mathfrak{sl}}(2)_k \oplus \hat{\mathfrak{sl}}(2)_{k'}$ representations. A natural class of $\hat{\mathfrak{sl}}(2|1)$ representations can be obtained by taking admissible representations of the two $\hat{\mathfrak{sl}}(2)$ algebras. However, by far not all of these $\hat{\mathfrak{sl}}(2|1)$ representations have been previously studied; in particular, the corresponding characters are only known for a subclass of these representations [9].

Characters of a broader class of $\hat{\mathfrak{sl}}(2|1)$ representations can be found by constructing resolutions; this in turn requires analyzing singular vectors in and the mappings between Verma modules. An important role is played here by spectral flows (twists) and other automorphisms of the $\hat{\mathfrak{sl}}(2|1)$ algebra. In this paper, we generalize the “continued” formulas for singular vectors [9], [10] in $\hat{\mathfrak{sl}}(2|1)$ Verma modules such that the new formulas are applicable to the case where degenerations of a certain type occur, under which the previously known formulas give the (incorrect) vanishing result. These are the degenerations where the so-called “charged” singular vector exists simultaneously with the MFF singular vectors to be introduced. This “stability” of singular vectors under degenerations of modules is ensured by incorporating twists into the “continued” formula. These singular vectors can be used to construct the resolutions and find character formulas.

2. $\hat{\mathfrak{sl}}(2|1)$ modules and automorphisms

2.1. The $\hat{\mathfrak{sl}}(2|1)$ algebra and automorphisms. The affine Lie superalgebra $\hat{\mathfrak{sl}}(2|1)$ is spanned by four bosonic currents $E^{12}$, $H^-$, $F^{12}$, and $H^+$, four fermionic ones, $E^1$, $E^2$, $F^1$, and $F^2$, and the

\footnotesize

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fermionic roots along the light-cone directions in Fig. 1. The root diagram of the finite-dimensional Lie superalgebra central element (which we identify with its eigenvalue and a family of automorphisms for \( u(1) \) subalgebra generated by \( H^+ \). The nonvanishing commutation relations are given by

\[
\begin{align*}
[E^1_m, F^2_n] &= F^1_{m+n}, & [H^m, E^1_{n,0}] &= -F^1_{m+n}, \\
[H^m, E^2_n] &= m\delta_{m+n,0}^k + 2H^-_{m+n}, & [H^m, H^\pm_n] &= \frac{1}{2}m\delta_{m+n,0}^k, \\
[F^1_m, E^2_n] &= F^1_{m+n}, & [E^1_{m+k}, F^2_n] &= -E^1_{m+n}, \\
[F^1_m, E^1_n] &= -F^2_{m+n}, & [E^2_{m+k}, F^1_n] &= E^2_{m+n}, \\
[H^m, E^1_n] &= \frac{1}{2}E^1_{m+n}, & [H^m, F^1_n] &= -\frac{1}{2}F^1_{m+n}, \\
[H^m, E^2_n] &= \frac{1}{2}E^2_{m+n}, & [H^m, F^2_n] &= \frac{1}{2}F^2_{m+n}, \\
[E^1_m, E^1_n] &= -m\delta_{m+n,0}^k + H^+_{m+n} - H^-_{m+n}, & [E^2_m, F^2_n] &= m\delta_{m+n,0}^k + H^+_{m+n} + H^-_{m+n}, \\
[E^1_m, E^2_n] &= E^1_{m+n}, & [F^1_m, F^2_n] &= F^1_{m+n}.
\end{align*}
\]

The Sugawara energy–momentum tensor is given by

\[
T_{\text{Sug}} = \frac{1}{k+1}(H^- H^- - H^+ H^+ + E^{12} F^{12} + E^1 F^1 - E^2 F^2).
\]

There are the algebra automorphisms

\[
\alpha: \begin{array}{cccc}
E^1_n & \mapsto & F^2_n, & F^2_n \mapsto E^1_n, \\
F^1_n & \mapsto & E^2_n, & F^1_{m+n} \mapsto E^1_n,
\end{array}
\] (2.3)

\[
\beta: \begin{array}{cccc}
E^1_n & \mapsto & F^1_n, & F^1_n \mapsto -E^1_n, \\
F^2_n & \mapsto & -E^2_n, & F^2_n \mapsto F^2_n
\end{array}. (2.4)
\]

and a family of automorphisms for \( \theta \in \mathbb{Z} \),

\[
\begin{array}{cccc}
E^1_n & \mapsto & E^1_{n-\theta}, & F^1_n \mapsto F^1_{n+\theta}, \\
E^2_n & \mapsto & E^2_{n+\theta}, & F^2_n \mapsto F^2_{n-\theta}, \\
H^+_n & \mapsto & H^+_n + k\theta \delta_{n,0}.
\end{array} (2.5)
\]