ESTIMATES FOR THE GHYS–LANGEVIN–WALCZAK
ENTROPY

ALEJANDRO M. MESÓN and FERNANDO VERICAT

ABSTRACT. In this note, we present a calculation of the Ghys–Langevin–Walczak entropy (GLW-entropy) by using the growth rate of periodic points. We also introduce a “functional” entropy for dynamics given by operators on functional spaces. It serves as a bound for the GLW-entropy.

1. Introduction

An entropy for dynamics given by group actions was introduced in [7]. In this paper the authors consider a compact space \((X, d)\) and a finitely generated pseudo-group of local homeomorphisms. They extend the classical concept of \((n, \varepsilon)\)-separated sets by using words of length \(n\). The Ghys–Langevin–Walczak entropy (GLW-entropy) is then defined by taking the growing rate of the \((n, \varepsilon)\)-separated sets with maximal cardinality. Accordingly, this entropy is a generalization of the Bowen classical entropy. This last one is recovered by taking the cyclic group generated by some homeomorphism \(T\), and the words being simply iterations of \(T\).

In this article, we consider a finitely generated group acting on a compact metric space \((X, d)\). A point \(x \in X\) is called \(n\)-periodic if there is a \(\gamma \in S(n)\) such that \(\gamma x = x\). Here, \(S(n) = \{\gamma \in \Gamma : \ell(\gamma) = n\}\) and \(\ell(\gamma)\) is the length of \(\gamma\), i.e., the minimal number of letters (generators of \(\Gamma\)) needed to express \(\gamma\). We establish a relation between the GLW-entropy and periodic points. We can also consider “periodic pseudo-orbits.” An \(\alpha\)-pseudo-orbit of a point \(x\), for a \(\Gamma\)-action on \(x\), is a mapping

\[\varphi_\varepsilon : \Gamma \to X, \text{ such that } d(\gamma x, \varphi_\varepsilon(\gamma)) \leq \alpha.\]

Biś and Walczak [4] proved that the GLW-entropy can be calculated by using pseudo-orbits. Based on this result we can relate the growth rates of

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periodic points and periodic pseudo-orbits. This could be useful for numerical estimations.

By "functional entropy" we mean the entropy whose structure includes, as a "phase space," an algebra of real or complex functions and whose dynamics are operators on these functions. An important contribution in this direction was supplied by Alicki, Fannes, and other authors [1]–[3]. The approach they used was physical, and the "quantity of information" was taken from the von Neumann entropy. The dynamics are implemented also by operators on partitions of the unity. Actually, the Alicki–Fannes entropy was designed for a general $C^*$-algebra, being the functional case a particular one.

In the present note, we introduce a modified version of the Alicki–Fannes entropy. It consists in implementing the dynamics by a finitely generated action group $\Gamma$. As it will be shown, this entropy is a lower bound for the $GLW$-entropy.

This paper is organized as follows. In the next section we consider the definition of the $GLW$-entropy and specify the main hypotheses whereas in Sec. 3 the main results are formulated and proved.

2. Preliminary definitions

Let $(X, d)$ be a compact metric space and $\Gamma$ be a finitely generated group acting on it. Recall that $S(n)$ denotes the sphere in $\Gamma$ centered at the identity of the radius $n$ in the word metric. We denote by $B(n)$ the ball of the radius $n$ in the word metric, i.e., $B(n) = \{\gamma \in \Gamma : \ell(\gamma) \leq n\}$. A subset $E$ of $X$ is $(n, \varepsilon)$-separated for the $\Gamma$-action if for every $x, y \in E$ ($x \neq y$) we have $d(\gamma x, \gamma y) > \varepsilon$, for some $\gamma \in B(n)$. Let

$$\beta_{n, \varepsilon} = \max \{\text{card } E : E \text{ is } (n, \varepsilon)\text{-separated}\},$$

and

$$h(\Gamma, \varepsilon) = \lim_{n \to \infty} \frac{1}{n} \log \beta_{n, \varepsilon}.$$

Then the $GLW$-entropy is defined by

$$h_{GLW}(\Gamma, X) = \lim_{\varepsilon \to 0} h(\Gamma, \varepsilon).$$

Next we declare our two main hypotheses:

**Expansivity.** We say that the group $\Gamma$ has an expansive action on $X$ if there exist a constant $\eta > 0$ such that for every $x, y \in X$ ($x \neq y$) there exists a positive integer $n$ with $d(\gamma x, \gamma y) > \eta$, for some $\gamma \in B(n)$. The number $\eta$ is called the constant of expansivity.