Abstract. Sárközy generalized a theorem of Erdös and Fuchs by considering sums of $k = 2$ members of $k = 2$ given “near” sequences. The aim of this paper is to extend this result for $k > 2$. We distinguish two cases according to the parity of $k$.

1. Introduction

Let $k \geq 2$ be a fixed integer and let $A^{(j)} = \{a^{(j)}_1, a^{(j)}_2, \ldots\}$ ($j = 1, \ldots, k$) be infinite sequences of integers such that $0 \leq a^{(j)}_1 \leq a^{(j)}_2 \leq \ldots$ ($j = 1, \ldots, k$). If $n$ is a non-negative integer, let $r_k(n)$ denote the number of solutions of

$$a^{(1)}_{i_1} + a^{(2)}_{i_2} + \ldots + a^{(k)}_{i_k} \leq n, \quad a^{(j)}_{i_j} \in A^{(j)} \quad (j = 1, \ldots, k).$$

Erdös and Fuchs [1] showed that if $c > 0$ then, for $k = 2$, $A^{(1)} = A^{(2)}$,

$$r_2(n) = cn + o(n^{1/4} \log^{-1/2} n)$$

cannot hold.

Sárközy [5] proved that if $c > 0$ and the sequences $A^{(1)}$ and $A^{(2)}$ are such as above and

$$a^{(2)}_i - a^{(1)}_i = o\left(\left( a^{(1)}_i \right)^{1/2} \log^{-1} a^{(1)}_i \right),$$

then (1) cannot hold. (A simple example shows that a condition of type (2) is necessary: Let $A^{(j)} = \left\{ \sum_1^k \varepsilon_i 2^{k+j} \right\}$, where $\varepsilon_i = 0$ or $1$ for $j = 1, \ldots, k$. Then $r_k(n) = n + 1$, thus $\lim_{n \to \infty} \frac{r_k(n)}{n} = \frac{1}{1 - O(1)}$.)

The object of this paper is to extend this result. We distinguish two cases according to the parity of $k$. The error term in our theorems is of smaller order of magnitude in the case when $k$ is odd (the difference between the

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error terms in these cases is only the exponent of $\log n$, if the main term is $cn$. We also investigate the case when $cn$ is replaced by $\sum_{b=1}^{s} c_{b} n^{\beta_{b}}$.

Note that our result is weaker than some known results for $A^{(1)} \equiv \ldots \equiv A^{(k)}$. Jurkat [4] showed in this case that if $k$ even, $0 < \beta \leq 1$ and $G(n) \sim cn^{\beta}$, then

$$r_{k}(n) = G(n) + o(n^{\beta/A})$$

cannot hold. Hayashi [2] proved that if $0 < \beta < 1$, $A^{(1)} \equiv A^{(2)} \equiv \ldots \equiv A^{(k)}$ and $G(n) - 2G(n-1) + G(n-2) \leq 0$ for all sufficiently large $n$, then for any $\varepsilon > 0$

$$r_{k}(n) = G(n) + o(n^{B-\varepsilon}),$$

where $B = \beta(1 - \beta)(1 - 1/k)/(1 - \beta + \beta/k)$, cannot hold. This improved Jurkat’s result if $\beta < \frac{3k-4}{3k-2}$. Let $\beta_{b}$ and $c_{b}$ $(b = 1, \ldots, s)$ be real numbers satisfying $\beta_{1} > \beta_{2} > \ldots > \beta_{s} \geq 0$ and $c_{1} > 0$. Hayashi [3] showed for $A^{(1)} \equiv \ldots \equiv A^{(k)}$ (and replaced $\sum_{b=1}^{s} c_{b} n^{\beta_{b}}$ by more general functions) that if $\frac{1}{2} < \beta_{1} < \frac{k}{2}$ then

$$(3) \quad r_{k}(n) = \sum_{b=1}^{s} c_{b} n^{\beta_{b}} + o(n^{\beta_{1}/2-1/A})$$

cannot hold.

2. The two main theorems

**Theorem 1.** Let $k \geq 2$ be a fixed even integer and let $A^{(j)} = \{a_{1}^{(j)}, a_{2}^{(j)}, \ldots\}$ $(j = 1, \ldots, k)$ be infinite sequences of integers such that $0 \leq a_{1}^{(j)} \leq a_{2}^{(j)} \leq \ldots \leq \ldots$ $(j = 1, \ldots, k)$. Let $\beta_{j}$ and $c_{j}$ $(j = 1, \ldots, s)$ be real numbers satisfying $\beta_{1} > \beta_{2} > \ldots > \beta_{s} \geq 0$ and $c_{1} > 0$.

(i) If $\frac{1}{2} < \beta_{1} < 1$ and

$$(4) \quad a_{i}^{(j+k/2)} - a_{i}^{(j)} = o\left(\left(\min\left(a_{1}^{(j)}, a_{2}^{(j+k/2)}\right)\right)^{1/2}\right) \quad \text{for} \quad j = 1, \ldots, \frac{k}{2},$$

then (3) cannot hold.

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