ON NONCOMPACT MINIMAL SETS OF THE GEODESIC FLOW

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ABSTRACT. We study nontrivial (i.e., containing more than one orbit) minimal sets of the geodesic flow on $\Gamma \backslash T^1\mathbb{H}^2$, where $\Gamma$ is a nonelementary Fuchsian group. It is not difficult to prove that nontrivial compact minimal sets always exist. We establish the existence of nontrivial noncompact minimal sets in two cases: (1) $\Gamma$ is a Schottky group of special kind generated by infinitely many hyperbolic elements, (2) $\Gamma$ contains a parabolic element (in particular, $\Gamma = \text{PSL}(2,\mathbb{Z})$). This is done by geometric coding of geodesic orbits and constructing a minimal set for symbolic dynamics with infinite alphabet.

INTRODUCTION

Let $\mathbb{H}^2$ be the upper half-plane $\{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ with the boundary $\partial\mathbb{H}^2 = \mathbb{R} \cup \{ \infty \}$. The group $G = \text{PSL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ by linear-fractional transformations and is the group of all orientation-preserving isometries of $\mathbb{H}^2$. Let $\Gamma$ be a Fuchsian group, i.e., a discrete subgroup of $G$. Then the space $\Gamma \backslash T^1\mathbb{H}^2 \simeq \Gamma \backslash G$ is the unit tangent bundle over the surface $M = \Gamma \backslash \mathbb{H}^2$ of constant negative curvature.

We study minimal sets of the geodesic flow $(\Gamma \backslash T^1\mathbb{H}^2, g_t)$, i.e., closed $g_\mathbb{R}$-invariant subsets of $\Gamma \backslash T^1\mathbb{H}^2$ that do not contain nonempty proper closed $g_\mathbb{R}$-invariant subsets. Trivial examples of such sets are given by periodic orbits and orbits that go to infinity in both directions. Periodic orbits exist for any nonelementary Fuchsian group $\Gamma$. In turn, closed noncompact orbits always exist unless $\Gamma$ is a cocompact lattice in $G$ (i.e., $M$ is compact). The first example of a nontrivial compact minimal set was given by Morse [7] in the 1920s. This was done by coding geodesic orbits and constructing a

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nontrivial minimal set for symbolic dynamics with the alphabet \(A = \{0, 1\}\) (see [6] for more details).

It is not difficult to prove that a nontrivial compact minimal set exists for any nonelementary Fuchsian group \(\Gamma\); see Lemma 3.3.

The existence of nontrivial noncompact minimal sets for the geodesic flow remained an open problem. This is in contrast to examples of such a minimal set for the horocycle flow \(\{h_s\}\) on \(\Gamma \backslash T^1 \mathbb{H}^2\). More precisely, if \(\Gamma\) is a convex-cocompact Fuchsian group which is not a lattice then the nonwandering set\(^1\) \(\Omega^+\) of the horocycle flow is noncompact, \(h_R\)-minimal, and \(g_R\)-invariant (hence nontrivial); see [8] for details.

Our study of minimal sets is partially motivated by a “classification” of minimal sets into 4 classes given in [9] for one-parameter homogeneous flows \((\Gamma \backslash G, g_t)\) on the homogeneous space \(\Gamma \backslash G\) of arbitrary Lie group \(G\). In connection with this result, it was not clear whether all these 4 classes occur for the geodesic flow on the modular surface. More precisely, in the classical case \(\Gamma = \text{PSL}(2, \mathbb{Z})\) it was not known whether there exist nontrivial noncompact minimal sets for the geodesic flow.

We formulate the following

**Conjecture.** Nontrivial noncompact minimal sets of the geodesic flow on \(\Gamma \backslash T^1 \mathbb{H}^2\) exist if and only if \(\Gamma\) is not a convex-cocompact Fuchsian group.

Note that nontrivial minimal sets are contained in the nonwandering subset \(\Omega \subset \Gamma \backslash T^1 \mathbb{H}^2\), and \(\Omega\) is compact if and only if \(\Gamma\) is convex-cocompact. Hence the necessity part of the conjecture is clear. We support the conjecture in the opposite direction by studying the following two classes of Fuchsian groups:

1. \(\Gamma\) is a Schottky group obtained by taking infinitely many disjoint semidisks in \(\mathbb{H}^2\) which accumulate to a single point in \(\partial \mathbb{H}^2\), and pairing them by hyperbolic isometries. Note that a detailed study of geodesic trajectories in this case was given in [5].

2. \(\Gamma\) contains a parabolic element (this includes, in particular, the modular group \(\text{PSL}(2, \mathbb{Z})\)). This case reduces to the situation where \(\Gamma\) is a Schottky group generated by one hyperbolic element \(h\) and one parabolic element \(p\) (see Sec. 4).

**Theorem.** If \(\Gamma\) is a Fuchsian group as in the cases (1)–(2), then there exists a nontrivial noncompact minimal set for the geodesic flow on \(\Gamma \backslash T^1 \mathbb{H}^2\).

\(^1\)The nonwandering subset \(\Omega\) for a continuous flow \(\phi_t\) on a topological space \(X\) consists of all points \(x \in X\) such that given any neighborhood \(O(x) \subset X\) there exists a sequence \(t_k \to \infty\) such that \(\phi_{t_k} O(x) \cap O(x) \neq \emptyset\) for all \(k\). Clearly, \(\Omega\) is a closed invariant subset of \(X\), and all orbits outside \(\Omega\) are locally closed.