Large Deviations of a Storage Process with Fractional Brownian Motion as Input

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[Received February 11, 2000; Revised June 25, 2001; Accepted June 26, 2001]

Abstract. We study probabilities of large extremes of the storage process \( Y(t) = \sup_{\sigma \geq t} (X(\sigma) - X(t) - c(\sigma - t)) \), where \( X(t) \) is the fractional Brownian motion. We derive asymptotic behavior of the maximum tail distribution for the process on fixed or slowly increased intervals by a reduction the problem to a large extremes problem for a Gaussian field.

Key words. extremes of Gaussian fields, storage process, fractional Brownian motion

AMS Subject Classification. Primary—60G70
Secondary—60G15

1. Introduction

We consider the storage process,

\[
Y(t) = \sup_{\sigma \geq t} (X(\sigma) - X(t) - c(\sigma - t)),
\]

where input process \( X(t) \) is the fractional Brownian motion (FBM) with Hurst parameter \( H, 0 < H < 1 \), that is a Gaussian centered a. s. continuous random process with stationary increments such that \( \mathbb{E}(X(t) - X(s))^2 = |t - s|^{2H} \). The constant \( c > 0 \) is called the service rate.

The FBM \( X(t) \) and corresponding process \( Y(t) \) where considered in Norros (1994) to model information traffic in telecommunication systems. Process \( Y(t) \) is stationary, its marginal tail distribution

\[
P\{Y(0) > u\} = P\left\{ \sup_{t \geq 0} X(t) - ct > u \right\}
\]

*Supported in part by the Stochastic Centre in Gothenburg, the Swiss National Science Foundation and RFFI Grants of Russian Federation 00-01-00247, 01-01-00649, 01-01-00644.
was studied by several authors, see for example Duffield and O’Connel (1996), Norros (1997), Narayan (1999). In Hüsler and Piterbarg (1999) and Narayan (1999) the asymptotic behavior of the probability as \( u \) tends to infinity has been derived. Narayan used Fourier representations of the Brownian motion and FBM and a similarity of theirs geometrical properties. Another approach, the Double Sum Method, is used in Hüsler and Piterbarg (1999). This approach is general for Gaussian processes and self-similarity property of the FBM only help to apply these general asymptotic methods. Hüsler and Piterbarg’s assumptions concerned only local stationary of \( X \) and behavior of its variance. There non-linear trends were considered as well.

The double sum method is also used in the present paper. One can find its essential description in Piterbarg (1996), see also references in that book and in Hüsler and Piterbarg (1999). We study the probability

\[
P\left\{ \sup_{t \in [0,T]} Y(t) > u \right\}
\]

(2)

for large \( u \) and fixed \( T \). This probability is equal to a probability of high extremes for a Gaussian two-parameter field which can be studied with double sum method. We derive asymptotic behavior of (2) as \( u \) tends to infinity. We found that the behavior is quite different for cases \( H > 1/2 \) and \( H < 1/2 \). Particularly, the behavior in the first case indicates existence of long overload periods in modeled telecommunication systems. For this connection see Norros (1999).

In Section 2 the auxiliary Gaussian field is introduced and its correlation structure and asymptotic properties of maximum distribution are studied. Section 3 contains main results of the present article on the storage process. The proofs of two technical lemmas, where double sum method is used, are given in Section 4.

The paper has originally been published as a technical report of Chalmers University, Piterbarg (2000). Patrik Albin has made an interesting observation soon after the publication—he noted that one of the assertions of Theorem 5 can be proven without using the double sum method, but by using the standard approach from the large extremes theory of sub-exponential processes. He had presented his simple and elegant solution immediately in a personal communication. It is my opinion that the readers would benefit from the opportunity to compare the two main methods for studying probability of large deviation of Gaussian and sub-exponential processes. The cases of \( H = 1/2 \) and \( H < 1/2 \) are not amenable to the line of attack proposed by Patrik. Proofs in these cases are based on all the auxiliary facts and results as presented in my original article. It precludes me from making substantial changes to the original manuscript. Patrik’s proof is included in the appendix with his gracious permission.