On the Maximum of Bivariate Normal Random Variables

ALAN P. KER

Department of Agricultural and Resource Economics, University of Arizona, PO Box 210023, Tucson, AZ 85721
E-mail: alan_ker@arizona.edu

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Abstract. The behavior of the mean and variance of the maximum of bivariate normal random variables to changes in the means, variances, and covariance of the underlying random variables is considered.

Key words. bivariate normal, monotonicity

1. Introduction

Extreme order statistics have been studied for the independent identically distributed case as well as for the independent non-identically distributed case (see David, 1981). With the exception of asymptotic approximations (see Reiss, 1989, for an introduction), little has been studied in the dependent case.

Let two real-valued random variables \( X \) and \( Y \) be bivariate normal with respective means and variances \( \mu_x, \mu_y, \sigma_x^2, \sigma_y^2 \), covariance \( \eta \), and correlation \( \rho = \eta / (\sigma_x \sigma_y) \). Define \( Z = \max[X, Y] \) and \( R = \min[X, Y] \).

Exact distributions of \( Z \) and \( R \) are useful in many applications including economic modeling, operations research, and genetics. The behavior of the mean and variance of \( Z \) and \( R \) with respect to changes in the means, variances, and covariance of \( X \) and \( Y \) has not previously been considered in the literature and is the purpose of this note. It is found that while much can be concluded about the behavior of \( E[Z] \) and \( E[R] \) to changes in \( \mu_x, \mu_y, \sigma_x^2, \sigma_y^2 \), and \( \eta \), nothing can be concluded about the behavior of \( \text{Var}[Z] \) and \( \text{Var}[R] \) to such changes.

For completeness note that the density of \( Z \)—derived by Basu and Ghosh (1978)—is

\[
f_Z(z) = \frac{1}{\sigma_y} \phi(A_y(z)) \Phi \left( \frac{A_y(z) - \rho A_x(z)}{\sqrt{1 - \rho^2}} \right) + \frac{1}{\sigma_x} \phi(A_x(z)) \Phi \left( \frac{A_x(z) - \rho A_y(z)}{\sqrt{1 - \rho^2}} \right)
\]

(1)
while using the moment generating function of $Z$—derived via Cain (1994)—produces the first two moments

$$E[Z] = \mu_x \Phi\left(\frac{\mu_x - \mu_y}{\theta}\right) + \mu_y \Phi\left(\frac{\mu_y - \mu_x}{\theta}\right),$$

and

$$E[Z^2] = (\sigma_x^2 + \mu_y^2)\Phi\left(\frac{\mu_x - \mu_y}{\theta}\right) + (\sigma_y^2 + \mu_x^2)\Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) + (\mu_x + \mu_y)\theta\phi\left(\frac{\mu_x - \mu_y}{\theta}\right)$$

where $\theta = (\sigma_x^2 - 2\eta + \sigma_y^2)^{1/2}$. Also note that since $E[Z]$ is differentiable and stochastically increasing in $\mu_x$ then $\partial E[Z]/\partial \mu_x > 0$. Similarly, $\partial E[Z]/\partial \mu_y > 0$, $\partial E[R]/\partial \mu_x < 0$, and $\partial E[R]/\partial \mu_y < 0$.

2. Derived results

**Result 1:** $\partial E[Z]/\partial \sigma_x^2 > 0$ and $\partial E[Z]/\partial \sigma_y^2 > 0$.

**Proof:** Note that $\partial E[Z]/\partial \sigma_x^2 = \partial E[Z]/\partial \theta \times \partial \theta/\partial \sigma_x^2$. It is easily seen that $\partial \theta/\partial \sigma_x^2 = \partial \theta/\partial \sigma_y^2 > 0$ (both inequalities assume $X$ and $Y$ have non-zero variance). What is less obvious is that $\partial E[Z]/\partial \theta > 0$.

$$\frac{\partial E[Z]}{\partial \theta} = \mu_x \Phi\left(\frac{\mu_x - \mu_y}{\theta}\right)\left(\frac{\mu_y - \mu_x}{\theta^2}\right) + \mu_y \Phi\left(\frac{\mu_y - \mu_x}{\theta}\right)\left(\frac{\mu_x - \mu_y}{\theta^2}\right)$$

$$+ \phi\left(\frac{\mu_y - \mu_x}{\theta}\right) + \delta\phi\left(\frac{\mu_y - \mu_x}{\theta}\right)
+ \phi\left(\frac{\mu_y - \mu_x}{\theta}\right)\left(\frac{\mu_x - \mu_y}{\theta^2}\right)$$

$$= \phi\left(\frac{\mu_x - \mu_y}{\theta}\right) + \phi\left(\frac{\mu_y - \mu_x}{\theta}\right)\left(\frac{\mu_x - \mu_y}{\theta^2}\right)\left(-\mu_x + \mu_y - \mu_y + \mu_y\right)$$

$$= \phi\left(\frac{\mu_x - \mu_y}{\theta}\right) > 0.$$

Given $\partial \theta/\partial \sigma_x^2 > 0$ and $\partial E[Z]/\partial \theta > 0 \rightarrow \partial E[Z]/\partial \sigma_x^2 > 0$. Similarly, given $\partial \theta/\partial \sigma_y^2 > 0$ and $\partial E[Z]/\partial \theta > 0 \rightarrow \partial E[Z]/\partial \sigma_y^2 > 0$. Using the same approach $\partial E[R]/\partial \theta < 0$ and thus $\partial E[R]/\partial \sigma_x^2 < 0$ and $\partial E[R]/\partial \sigma_y^2 < 0$ are easily shown.

**Result 2:** $\partial E[Z]/\partial \eta < 0$. 