Euler–Poincaré Pairings and Elliptic Representations of Weyl Groups and p-Adic Groups

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Abstract. The space of elliptic virtual representations of a p-adic group is endowed with a natural inner product $EP(\ , \ )$, defined analytically by Kazhdan and homologically by Schneider–Stuhler. Arthur has computed $EP$ in terms of analytic $R$-groups. For Iwahori spherical representations, we show that $EP$ can also be expressed in terms of a corresponding inner product on space of elliptic virtual representations of Weyl groups. This leads to an explicit description of both elliptic representation theories, in terms of the Kazhdan–Lusztig and Springer correspondences.


Key words. elliptic representations, Weyl groups, p-adic groups.

1. Introduction

Let $G$ be a connected split adjoint group over a non-Archimedean local field $F$ of arbitrary characteristic. Schneider and Stuhler have defined a pairing $EP(V, V')$ between admissible representations of $G(F)$ by the formula

$$EP(V, V') = \sum_{n \geq 0} (-1)^n \dim \text{Ext}^n(V, V'),$$

where Ext is taken in the category of smooth representations of $G(F)$, and they prove that $EP(V, V')$ is the trace on $V'$ of a certain function $f_V$ on $G(F)$. They also show, assuming the characteristic of $F$ is zero, that $EP(V, V')$ equals the elliptic inner product of the characters of $V, V'$. (See [K] for background on elliptic representation theory.)

On the other hand, Arthur [A] has calculated the elliptic inner product in terms of elliptic characters of the analytic $R$-group $R_{an}$ (‘analytic’ because it is defined by zeros of Plancherel measures). Arthur then suggests that his formula “. . . might also play a role in the general character theory of Weyl groups”. One purpose of this paper is to confirm this prediction: We show that the Iwahori-spherical elliptic

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representation theory of $G(F)$ is equivalent to the elliptic representation theory of the Weyl groups of endoscopic groups of $G$.

In addition, this paper contains the following.

(1) We describe explicitly the elliptic virtual representations of a Weyl group $W$, in terms of Springer representations, using a Weyl group analogue of the pairing $EP$.

(2) We describe explicitly the elliptic Iwahori-spherical virtual representations of $G(F)$, in terms of Kazhdan–Lusztig parameters. This is in the same spirit as the classification of tempered and discrete series representations in [KL].

(3) For Iwahori spherical representations, Arthur’s formula now holds in any characteristic, provided we use the homological definition of $EP$, instead of the elliptic inner product of characters.

(4) In order to apply Arthur’s formula to the Kazhdan–Lusztig correspondence, we describe the analytic $R$-group $R_{an}$, and the cocycle $\eta_{an}$ (arising in Harish-Chandra’s theory of intertwining operators) in terms of a geometric $R$-group and cocycle attached to the Kazhdan–Lusztig parameter.

To give a more precise exposition, we begin with Weyl groups. Let $\hat{G}$ be a simply-connected Lie group with Weyl group $W$. Then $W$ has an elliptic representation theory, in which proper Levi subgroups are stabilizers of nonzero vectors in the reflection representation $E$ of $W$. Let $\mathcal{R}(W)$ be the span of the irreducible representations of $W$, and let $\tilde{\mathcal{R}}(W)$ be the quotient of $\mathcal{R}(W)$ by the span of all induced representations from proper Levi subgroups. We define an analogue of the pairing $EP$ on $\tilde{\mathcal{R}}(W)$ as follows:

$$e_W(\chi, \chi') = \sum_{n \geq 0} (-1)^n \dim \text{Hom}_W(A^n E \otimes \chi, \chi').$$

This pairing is initially defined on $\mathcal{R}(W)$, but its radical is exactly the kernel of the map $\mathcal{R}(W) \to \tilde{\mathcal{R}}(W)$, hence $e_W$ is a nondegenerate pairing on $\tilde{\mathcal{R}}(W)$.

For $x \in \hat{G}$, let $A_x$ be the component group of the centralizer of $x$ in the adjoint group of $\hat{G}$. The groups $A_x$ also have ‘Levi’ subgroups, hence their own elliptic representation theories, hence pairings $e_{A_x}$ analogous to $e_W$.

Let $\mathcal{U}_G$ be the set of unipotent elements in $\hat{G}$, modulo conjugacy. Combining results of Borho–MacPherson, Lusztig and Springer, we have a decomposition $\mathcal{R}(W) = \bigoplus_{\text{regular}} \mathcal{R}_u(W)$, together with an isomorphism $H_u : \mathcal{R}_u(A_u) \to \mathcal{R}_u(W)$, where $H_u(\rho)$ is the Springer representation of $W$ on $\text{Hom}_A(\rho, H(B_u))$, $B_u$ denotes the fixed points of $u$ in the flag manifold $B$ of $\hat{G}$, $H(B_u)$ is the cohomology of $B_u$, with grading ignored, and $\mathcal{R}_u(A_u)$ is the span of the irreducible representations of $A_u$ which appear in the natural action of $A_u$ on $H(B_u)$. The representations $H_u(\rho)$ have been calculated explicitly in [BS], [Sho1] for $W$ of exceptional type,