K-Functionals and Exact Values of \( n \)-Widths of Certain Classes in the Spaces \( C(2\pi) \) and \( L_1(2\pi) \)

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Received July 7, 1999; in final form, July 27, 2001

Abstract—For classes of \( 2\pi \)-periodic functions whose \( K \)-functionals are majorized by functions satisfying certain constraints, exact values of Kolmogorov, Bernstein, and trigonometric \( n \)-widths in the spaces \( C(2\pi) \) and \( L_1(2\pi) \) are obtained. Examples of majorants that satisfy the requirements stated in this paper are given.

Key words: periodic Lebesgue \( p \)th power integrable function, \( K \)-functional, Kolmogorov \( n \)-width, Bernstein \( n \)-width, \( 2\pi \)-periodic measurable function.

1. Let us denote by \( L_p(2\pi) \) (\( 1 \leq p < \infty \)) the space of \( 2\pi \)-periodic \( p \)th power Lebesgue integrable functions \( g(x) \) on \([0, 2\pi]\) with norm

\[
\|g\|_p = \left\{ \int_0^{2\pi} |g(x)|^p \, dx \right\}^{1/p},
\]

and by \( L_\infty(2\pi) \) the space of \( 2\pi \)-periodic measurable, essentially bounded (on the whole axis) functions \( g(x) \) with norm

\[
\|g\|_\infty = \sup \{|g(x)| : 0 \leq x \leq 2\pi\}.
\]

By \( C(2\pi) \) we mean the space of continuous \( 2\pi \)-periodic functions \( g(x) \) with the uniform norm

\[
\|g\|_C = \max \{|g(x)| : 0 \leq x \leq 2\pi\}.
\]

Obviously, if \( g(x) \) is a continuous function, then \( \|g\|_\infty = \|g\|_C \). By \( L^r_p(2\pi) \) (\( r \in \mathbb{N}, 1 \leq p \leq \infty \)) we denote the set of functions \( f(x) \) with period \( 2\pi \) defined on the whole axis whose \( (r-1) \)st derivative \( f^{(r-1)}(x) \) is absolutely continuous on \([0, 2\pi]\) and \( f^{(r)}(x) \in L_p(2\pi) \).

For \( f(x) \in L_p(2\pi) \) (\( 1 \leq p < \infty \)), the \( K \)-the functional \( K_{r,p}(f, t) \) (\( r \in \mathbb{N}, t > 0 \)) is defined as follows (see, for example, [1]):

\[
K_{r,p}(f, t) = \inf \{ \|f - g\|_p + t\|g^{(r)}\|_p : g(x) \in L^r_p(2\pi) \}.
\]

In the case \( f(x) \in C(2\pi) \), we have

\[
K_{r,\infty}(f, t) = \inf \{ \|f - g\|_C + t\|g^{(r)}\|_\infty : g(x) \in L^r_\infty(2\pi) \}.
\]

This characteristic indicates how well the function \( f(x) \) can be approximated by the smooth functions \( g(x) \) with the value of \( \|g^{(r)}\|_p \) taken into account. It is well known [1] that there exist
constants \( \tilde{c}_{1,p,r}, \tilde{c}_{2,p,r} \) depending only on \( r,p \) and such that if \( f(x) \in L_p(2\pi) \) (or \( f(x) \in C(2\pi), p = \infty \)), then

\[
\tilde{c}_{1,p,r} \omega_{r,p}(f, t) \leq K_{r,p}(f, t') \leq \tilde{c}_{2,p,r} \omega_{r,p}(f, t), \quad t > 0,
\]

where \( \omega_{r,p}(f, t) = \sup\{\|\Delta_h^r(f, x)\|_p : 0 < h \leq t\} \) is the modulus of smoothness of \( r \)th order of the function \( f(x) \) and \( \Delta_h^r(f, x) = \sum_{j=0}^{r-1} (-1)^{r-j} \binom{r}{j} f(x + jh) \) is the \( r \)th finite difference with mesh \( h \) of the function \( f(x) \) at the point \( x \).

To define a \( K \)-functional of special form, let us denote by \( V(2\pi) \) the set of \( 2\pi \)-periodic functions \( f(x) \) of bounded variation on the closed interval \([0, 2\pi]\) and by \( V^r(2\pi) \) (\( r \in \mathbb{N} \)) the set of \( 2\pi \)-periodic functions \( f(x) \) whose derivative \( f^{(r-1)}(x) \) is absolutely continuous and \( f^{(r)}(x) \in V(2\pi) \). Then for \( f(x) \in L_1(2\pi) \), in addition to the \( K \)-functional \( K_{r,1}(f, t) \), we also consider the \( K \)-functional

\[
K_{r,V}^*(f, t) = \inf \left\{ \|f - g\|_1 + t \int_0^{2\pi} (g^{(r)}) : g(x) \in V^r(2\pi) \right\}, \quad t > 0,
\]

where \( \int_0^{2\pi} (g^{(r)}) \) is the variation of the function \( g^{(r)}(x) \) on the closed interval \( 0 \leq x \leq 2\pi \). For \( r = 0 \), we assume in what follows that

\[
K_{0,V}^*(f, t) = \left\{ \|f - g\|_1 + t \int_0^{2\pi} (g) : g(x) \in V(2\pi) \right\}, \quad t > 0.
\]

2. Let us show that the \( K \)-functional \( K_{r,V}^*(f, t) \) satisfies a relation of the form (1).

**Lemma 1.** There exist constants \( c_{1,r} > 0 \) and \( c_{2,r} > 0 \) depending only on \( r \) \((r = 0, 1, 2, \ldots)\) and such that if \( f(x) \in L_1(2\pi) \), then

\[
c_{1,r} \omega_{r+1,1}(f, t) \leq K_{r,V}^*(f, t^{r+1}) \leq c_{2,r} \omega_{r+1,1}(f, t), \quad t > 0.
\]

**Proof.** Taking note of the inclusion \( L_1^{r+1}(2\pi) \subset V^{r}(2\pi) \), which follows from the relation

\[
\int_0^{2\pi} (g^{(r)}) = \int_0^{2\pi} |g^{(r+1)}(x)| dx \quad \forall g(x) \in L_1^{r+1}(2\pi),
\]

we can write

\[
K_{r,V}^*(f, t) \leq \inf \{ \|f - g\|_1 + t \int_0^{2\pi} (g) : g(x) \in L_1^{r+1}(2\pi) \} = K_{r+1,1}(f, t).
\]

Using (4) and (1), we obtain the upper bound

\[
K_{r,V}^*(f, t^{r+1}) \leq \tilde{c}_{2,1,r+1} \omega_{r+1,1}(f, t).
\]

To obtain a lower bound, we need the well-known inequality

\[
\|\Delta_h^{r+1}(\varphi, x)\|_1 \leq h^{r+1} \|\varphi^{(r+1)}\|_1 \quad \forall \varphi(x) \in L_1^{r+1}(2\pi).
\]

To each function \( g(x) \in V^{r}(2\pi) \) we assign the Steklov function

\[
g_{\nu}(x) = \frac{1}{2\nu} \int_{x-\nu}^{x+\nu} g(x + t) dt, \quad \nu > 0.
\]