THUE INEQUALITIES WITH A SMALL NUMBER
OF PRIMITIVE SOLUTIONS

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Dedicated to Professor András Sárközy on the occasion of his 60th birthday

Abstract

Several upper bounds are known for the numbers of primitive solutions \((x, y)\) of the Thue equation (1) \(|F(x, y)| = m\) and the more general Thue inequality (3) \(0 < |F(x, y)| \leq m\). A usual way to derive such an upper bound is to make a distinction between “small” and “large” solutions, according as \(\max(|x|, |y|)\) is smaller or larger than an appropriate explicit constant \(Y\) depending on \(F\) and \(m\); see e.g. [1], [11], [6] and [2]. As an improvement and generalization of some earlier results we give in Section 1 an upper bound of the form \(cn\) for the number of primitive solutions \((x, y)\) of (3) with \(\max(|x|, |y|) \geq Y_0\), where \(c \leq 25\) is a constant and \(n\) denotes the degree of the binary form \(F\) involved (cf. Theorem 1). It is important for applications that our lower bound \(Y_0\) for the large solutions is much smaller than those in [1], [11], [6] and [4], and is already close to the best possible in terms of \(m\). By using Theorem 1 we establish in Section 2 similar upper bounds for the total number of primitive solutions of (3), provided that the height or discriminant of \(F\) is sufficiently large with respect to \(m\) (cf. Theorem 2 and its corollaries). These results assert in a quantitative form that, in a certain sense, almost all inequalities of the form (3) have only few primitive solutions. Theorem 2 and its consequences are considerable improvements of the results obtained in this direction in [3], [6], [13] and [4]. The proofs of Theorems 1 and 2 are given in Section 3. In the proofs we use among other things appropriate modifications and refinements of some arguments of [1] and [6].

1. The number of large primitive solutions

Let \(F(X, Y) \in \mathbb{Z}[X, Y]\) be an irreducible binary form of degree \(n \geq 3\), and let \(m\) denote a positive integer with \(s\) distinct prime factors. First consider the primitive solutions of the Thue equation

\[ |F(x, y)| = m \quad \text{in} \quad x, y \in \mathbb{Z}, \]

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that is the solutions in coprime integers \( x, y \). Solutions \((x, y)\) and \((-x, -y)\) will be regarded as the same.

Bombieri and Schmidt [1] derived an upper bound of the form \( cn^{s+1} \) for the number of primitive solutions of (1), where \( c > 0 \) is an absolute constant. For \( m = 1 \) this is best possible, except for the determination of \( c \). It was proved in [1] by induction on \( s \) that if \( N_n \) denotes the corresponding bound in the special case \( m = 1 \), one obtains \( N_n n^s \) as a bound in the general case. To derive bound for \( m = 1 \), the authors drew a distinction between “large” and “small” solutions. In case of large solutions they combined the Thue–Siegel principle with a strong gap principle.

For large \( n \), Bombieri and Schmidt [1] established their bound with \( c = 215 \). Later Stewart [13] derived the bound \( 2800 n^{s+1} \) for every \( n \geq 3 \), and improved it to \( 4n^s \) for sufficiently large \( m \).

Recently Brindza, Pintér, van der Poorten and Waldschmidt [2] have obtained an upper bound for the number of large primitive solutions of (1). They proved that (1) has at most \( 2n^s(s + 1) + 13n \) primitive solutions \((x, y)\) with

\[
H(x, y) \geq 21n^2 M^s m^{\frac{1}{m} \cdot (s+1)}
\]

where \( M = M(F) \) denotes the Mahler height of \( F \) (see below) and \( H(x, y) = \max(|x|, |y|) \). Their proof is based on Baker’s theory of linear forms in logarithms and a new gap principle.

In what follows, we shall be concerned with the primitive solutions of the Thue inequality

\[
0 < |F(x, y)| \leq m \quad \text{in} \quad x, y \in \mathbb{Z}
\]

where \( \gcd(x, y) = 1 \) and the solutions \((x, y)\) and \((-x, -y)\) are again deemed as the same. It follows from a result of Mahler [9] that (3) has at most \( c(F)m^{2/n} \) primitive solutions, where \( c(F) \) depends only on \( F \). In this bound the dependence on \( m \) is best possible if \( m \) is large. Recently Thunder [14] showed that \( c(F) \) can be replaced by a constant depending only on \( n \). In fact Mahler and Thunder established their results for all, not necessarily primitive solutions \((x, y)\) of (3).

Much better upper bounds can be obtained for the number of primitive solutions of (3) if we restrict ourselves to the large solutions or if we make some assumptions on \( F \). Further, it is not necessary to assume that \( F \) is irreducible. From now on we assume only that, unless otherwise stated, the discriminant \( D(F) \) of \( F \) is different from zero. To state results of this type we introduce some further notation and terminology.

For \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \), the binary forms \( F(X, Y) \) and \( F_A(X, Y) := F(ax + by, cx + dy) \) will be called equivalent. Neither \( D(F) \), nor the number of primitive solutions of (3) changes if \( F \) is replaced by an equivalent form. Each binary form is equivalent to one with \( F(1, 0) \neq 0 \). We assume that

\[
F(X, Y) = a_0(X - \alpha_1 Y) \cdots (x - \alpha_n Y)
\]