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Topos Perspective on the Kochen–Specker Theorem: IV. Interval Valuations

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We extend the topos-theoretic treatment given in previous papers (Butterfield, J. and Isham, C. J. (1999). *International Journal of Theoretical Physics* **38**, 827–859; Hamilton, J., Butterfield, J., and Isham, C. J. (2000). *International Journal of Theoretical Physics* **39**, 1413–1436; Isham, C. J. and Butterfield, J. (1998). *International Journal of Theoretical Physics* **37**, 2669–2733) of assigning values to quantities in quantum theory. In those papers, the main idea was to assign a sieve as a partial and contextual truth value to a proposition that the value of a quantity lies in a certain set $\Delta \subseteq \mathbb{R}$. Here we relate such sieve-valued valuations to valuations that assign to quantities subsets, rather than single elements, of their spectra (we call these "interval" valuations). There are two main results. First, there is a natural correspondence between these two kinds of valuation, which uses the notion of a state's support for a quantity (Section 3). Second, if one starts with a more general notion of interval valuation, one sees that our interval valuations based on the notion of support (and correspondingly, our sieve-valued valuations) are a simple way to secure certain natural properties of valuations, such as monotonicity (Section 4).

KEY WORDS: Kochen–Specker theorem; topos theory; valuations; supports; quantum logic.

1. INTRODUCTION

In three previous papers (Butterfield and Isham, 1999; Hamilton *et al.*, 2000; Isham and Butterfield, 1998) we have developed a topos-theoretic perspective on the assignment of values to quantities in quantum theory. In particular, it was shown that the Kochen–Specker theorem (Kochen and Specker, 1967) (which states the impossibility of assigning to each bounded self-adjoint operator on a Hilbert space of dimension greater than 2, a real number such that functional relations are preserved) is equivalent to the nonexistence of any global elements of

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a certain presheaf Σ, called the "spectral presheaf." This presheaf is defined in closely analogous ways on the category $\mathcal{O}$ of bounded self-adjoint operators on a Hilbert space $\mathcal{H}$ (cf. Butterfield and Isham, 1999; Isham and Butterfield, 1998), and on the category $\mathcal{V}$ of commutative von Neumann subalgebras of the algebra of bounded operators on $\mathcal{H}$ (cf. Hamilton *et al.*, 2000).[4]

A key result of Butterfield and Isham (1999), Hamilton *et al.* (2000), and Isham and Butterfield (1998) is that, notwithstanding the Kochen–Specker theorem, it *is* possible to define "generalized valuations" on all quantities, in which any proposition "$A \in \Delta$" (read as saying that the value of the physical quantity A lies in the Borel set of real numbers Δ) is assigned, in effect, a set of quantities that are coarse-grainings (functions) of A. To be precise, such a proposition is assigned as a truth value a certain set of morphisms in the category $\mathcal{O}$ (or $\mathcal{V}$), this set being required to have the structure of a *sieve*. These generalized valuations can be motivated from various different perspectives (cf. also Butterfield, 2001). In particular, they obey a condition analogous to the *FUNC* condition of the Kochen–Specker theorem, which states that assigned values preserve functional relations between operators, and certain other natural conditions too. Furthermore, each (pure or mixed) quantum state defines such a valuation. In Section 2 we will briefly recall the details of these proposals and results.

In this paper, we shall extend this treatment in two main ways (Sections 3 and 4 respectively). Both involve the relation between sieve-valued valuations and valuations that assign to a quantity A, not an individual member of its spectrum, but rather some subset of it (which we call "interval valuations"). Though this idea seems at first sight very different from our generalized valuations—that assign sieves to propositions "$A \in \Delta$"—the two types of valuations turn out to be closely related. In fact, there is a natural correspondence between them that uses the notion of the *support* of a state for a quantity (Section 3). This correspondence is best expressed for the case of $\mathcal{V}$ than for $\mathcal{O}$ since, by using von Neumann algebras as the base category, various measure-theoretic technicalities can be immediately dealt with. However, we shall also discuss the case of $\mathcal{O}$, as it is heuristically valuable.

In Section 4, we describe how if one starts with a yet more general notion of an interval valuation (i.e., one that does not appeal to the notion of support), one sees that our interval valuations based on the notion of support (and correspondingly, our sieve-valued valuations) are a simple way to secure certain natural properties of valuations, such as monotonicity.

[4] There is also a closely analogous presheaf—the dual presheaf **D**—that is defined on the category $\mathcal{W}$ of Boolean subalgebras of the lattice $\mathcal{L}(\mathcal{H})$ of projectors on $\mathcal{H}$; and the Kochen–Specker theorem is also equivalent to the nonexistence of any global elements of **D** (cf. Butterfield and Isham, 1999; Isham and Butterfield , 1998). But we shall not discuss $\mathcal{W}$ further in this paper.