Bounds on Gambler’s Ruin Probabilities in Terms of Moments

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Abstract. Consider a wager that is more complicated than simply winning or losing the amount of the bet. For example, a pass line bet with double odds is such a wager, as is a bet on video poker using a specified drawing strategy. We are concerned with the probability that, in an independent sequence of identical wagers of this type, the gambler loses L or more betting units (i.e., the gambler is “ruined”) before he wins W or more betting units. Using an idea of Markov, Feller established upper and lower bounds on the probability of ruin, bounds that are often very close to each other. However, his formulation depends on finding a positive nontrivial root of the equation \( \phi(\rho) = 1 \), where \( \phi \) is the probability generating function for the wager in question. Here we give simpler bounds, which rely on the first few moments of the specified wager, thereby making such gambler’s ruin probabilities more easily computable.

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1. Introduction

Let \( X \) be an integer-valued random variable representing the profit from a gambling opportunity, in betting units (positive, negative, and zero values correspond respectively to a win, loss, and tie for the gambler). We assume that

\[
P\{-\nu \leq X \leq \mu\} = 1, \quad P\{X = -\nu\} > 0, \quad P\{X = \mu\} > 0,
\]

where \( \mu \) and \( \nu \) are positive integers, and that

\[
E[X] \neq 0.
\]

Letting

\[
\phi(\rho) := E[\rho^X]
\]

denote the probability generating function, we note that \( \phi(1) = 1, \phi'(1) = E[X] \), and, by (1.1), \( \phi(\rho) > 1 \) for sufficiently small \( \rho \in (0, 1) \) and for sufficiently large \( \rho \in (1, \infty) \). Since \( X(X - 1) \geq 0 \), it follows that \( \phi \) is convex on \( (0, \infty) \), and so there exists a unique \( \rho_0 \in (0, 1) \cup (1, \infty) \) such that
\[ \phi(\rho_0) = 1. \]  
(1.4)

If \( E[X] < 0 \), then \( \rho_0 > 1 \). If \( E[X] > 0 \), then \( \rho_0 < 1 \).

Now let \( X_1, X_2, \ldots \) be independent and identically distributed (i.i.d.) with common distribution that of \( X \), representing the profits from repeated independent trials of the given gambling opportunity. Then

\[ S_n := X_1 + \cdots + X_n \]

represents the gambler’s profit after \( n \) such trials. We suppose that the gambler’s final bet is on trial

\[ N(-L, W) := \min \{ n \geq 1 : S_n \leq -L \quad \text{or} \quad S_n \geq W \}, \]

(1.6)

where \( W \) and \( L \) are positive integers, that is, he stops betting as soon as he wins at least \( L \) betting units or loses at least \( L \) betting units.

Following Markov (1912) and Uspeński (1937), Feller (1950) obtained bounds on the probability of ruin \( \rho \), or, equivalently, the probability of success:

\[ \frac{\rho_0^L - 1}{\rho_0^{L+W+\mu-1} - 1} \leq P\{S_{N(-L, W)} \geq W\} \leq \frac{\rho_0^{L+\nu-1} - 1}{\rho_0^{L+W+\nu-1} - 1}. \]

(1.7)

(See Feller (1968), Equation (8.12) of Chapter XIV.) In particular, if \( P\{X = +1\} = p > 0 \), \( P\{X = -1\} = q > 0 \), and \( P\{X = 0\} = r \geq 0 \), where \( p + q + r = 1 \) and \( p \neq q \), then \( \rho_0 = q/p \) and \( \mu = \nu = 1 \), so (1.7) becomes

\[ P\{S_{N(-L, W)} = W\} = \frac{(q/p)^L - 1}{(q/p)^{L+W} - 1}, \]

(1.8)

which is, of course, well known.

Uspeński (1937) treated the special case in which \( P\{X = -\nu\} + P\{X = \mu\} = 1 \). However, his formulation required that the gambler avoid overshooting the boundaries \( -L \) and \( W \). This is natural if one assumes, as did Uspeński, that initially the gambler has \( L \) units and his opponent has \( W \) units. On the other hand, Feller’s model can be regarded as that of a gambler (perhaps in a casino) whose goal is to win \( W \) or more units before losing \( L \) or more units. We prefer the latter approach.

Notice that \( \rho_0 \) is a root of a polynomial of degree \( \mu + \nu \), so, except in a few special cases, some numerical scheme is usually needed to evaluate \( \rho_0 \). Furthermore, when \( \rho_0 \) is close to 1, as it frequently is, one may need to use high-precision arithmetic to prevent serious rounding error.

Our aim in this paper is to find bounds on the success probability \( P\{S_{N(-L, W)} \geq W\} \), expressible solely in terms of the first four moments of \( X \) (and of course \( W, L, \mu, \) and \( \nu \)), that are often nearly as accurate as those in (1.7) and much easier to compute. We also bound \( E[N(-L, W)] \), the expected duration of the session, in a similar way.

In addition, we relax Feller’s assumptions a bit, still requiring that \( X \) be bounded, but no longer requiring that \( X \) be integer-valued or even discrete. The point is that, if a game has fractional payoffs (see Section 4 for examples of this), one should not be forced to rescale the basic monetary unit in order to apply these results.

Our original goal was to estimate the error in an approximation to the gambler’s ruin