STUDY OF EXPPLICIT ANALYTIC SOLUTIONS FOR THE NONLINEAR COUPLED SCALAR FIELD EQUATIONS

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Abstract: By using two different transformations, several types of exact analytic solutions for a class of nonlinear coupled scalar field equation are obtained, which contain soliton solutions, singular solitary wave solutions and triangle function solutions. These results can be applied to other nonlinear equations. In addition, parts of conclusions in some references are corrected.

Key words: nonlinear coupled scalar field equation; analytic solution; soliton solution; periodic solution

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Introduction

With the rapid development of nonlinear science, many phenomena in physics, mechanics, chemistry and biology etc. can be described simply and exactly by the mathematical model—nonlinear equations\textsuperscript{[1-7]}. On the contrary, in order to study these phenomena quantitatively. It is very important to find explicit and exact solutions of these nonlinear equations\textsuperscript{[1-7]}. However, there is no unified model to solve nonlinear equations up-to-date. Therefore, finding more and more effective methods is still an important project.

In this paper, we consider the following nonlinear coupled scalar field equations\textsuperscript{[3-6]}:
\begin{align}
\sigma_{xx} &= -\sigma + \sigma^3 + d\rho^2 \sigma, \\
\rho_{xx} &= f\rho + \lambda \rho^3 + d\rho(\sigma^2 - 1),
\end{align}
where $\sigma$ and $\rho$ are real scalar fields, $d$, $f$ and $\lambda$ are parameters. Eq. (1a) mainly came from studying of quantization hoy electric soliton in basic particle theory\textsuperscript{[3]}. Eq. (1a) had been studied by many authors\textsuperscript{[3-6]}. Rajaraman\textsuperscript{[3-6]} obtained the following form soliton solution of Eq. (1a) by using the orbit function method
\begin{align}
0 < f < 1/2, \quad \lambda = (d - f)^2, \quad d > 0, \\
\sigma &= \pm \tanh[\sqrt{f}(x + x_0)], \quad \rho = \pm \sqrt{(1 - 2f)/d}\sech[\sqrt{f}(x + x_0)].
\end{align}

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In fact, the conditions of the kind of solution (2) are not right. They would be changed into \( f, (1 - 2f) d > 0 \), \( \lambda = d(d - 2f)/(1 - 2f) \). Thus the solution (3) is the exact solution for Eq. (1). Eq. (1) was studied by using the following transformation (4) in Ref. [3]

\[
\rho = \sqrt{\frac{2d - 3}{d - 1}} \sigma, \quad \sigma_z = \sigma \sqrt{(d - 1) \sigma^2 + \frac{(2d - 3)^2}{d^2(2 - d)}}
\]

\[
\left\{ \begin{array}{l}
  f = d - 1, \\
  \lambda = \frac{d(d - 2)}{2d - 3}
\end{array} \right.
\]

(4)

In fact the transformation (4) is also error. It should be changed into \( \rho = \sqrt{(2d - 3)/d} \sigma, \sigma_z = \sigma \sqrt{(d - 1) \sigma^2 - 1} \). Therefore, these soliton solutions of \( \sigma \) and \( \rho \) with bell-type (sech \( x \)) via using the transformation (4) are not right, and it is directly shown that Eq. (1) has not the soliton solution that \( \sigma \) and \( \rho \) are both bell-type (sech \( x \)). Therefore, the singular solitary wave solution and period solution of Eq. (1) obtained in Ref. [6] are also wrong by the use of the transformation (4).

In this paper, we would like to consider Eq. (1) from two angles: firstly based on the transformation[31], through using a series of ansatzes, several exact analytic solutions for Eq. (1) are found, which contain soliton solutions, singular solitary solutions and triangle function solutions. Some of them prove that the forecast in the Ref. [6] (Eq. (1) has the soliton solution of \( \sigma \) and \( \rho \) are both of kink-type (tanh \( x \))) is sure. Then by using new transformation, we find sine and cosine form solutions. Finally, by using new transformations, we derive other triangle function solutions which contain ones that the author in Ref. [6] wanted to seek for. In addition, these conclusions can be applied to other equations in physics and mechanics etc.

1 Exact Analytic Solutions for Eq. (1)

For the given Eq. (1), taking the following transformation\[^3,4\]

\[
\sigma_x = A + B \sigma^2 + C \rho^2, \quad \rho = D \sigma,
\]

(5)

where \( A, B, C \) and \( D \) are constants to be determined later. We would like to consider Eq. (1) via using the transformation (5). Substituting Eq. (5) into Eqs. (1a) and (1b) respectively, by comparing the coefficients, yields the following relations

\[
2AB = -1, \quad 2B^2 = 1, \quad 2BC + 2CD = d,
\]

(6)

\[
f - d = AD, \quad \lambda = DC, \quad D^2 + BD = d.
\]

(7)

Solving Eqs. (6) and (7) yields

\[
A = -\frac{1}{\sqrt{2}} \epsilon, \quad B = \frac{1}{\sqrt{2}} \epsilon, \quad C = \epsilon \mu \sqrt{\frac{\lambda}{2}}, \quad D = \epsilon \mu \sqrt{2 \lambda},
\]

(8)

\[
d = 2\lambda + \mu \sqrt{\lambda}, \quad f = 2\lambda \quad (\lambda > 0),
\]

(9)

where \( \epsilon = \pm 1, \mu = \pm 1 \). Combining (6) – (8) and the transformation (5), we have the following formulations

\[
\sigma = \epsilon \mu \frac{1}{\sqrt{2\lambda}} \frac{\rho}{\rho},
\]

(10)

\[
\rho_{xx} - \frac{d}{2\lambda} \frac{\rho_x^2}{\rho} - \mu \sqrt{\lambda} \rho - \lambda \rho^3 = 0.
\]

(11)

Taking another transformation \( \rho_x = \sqrt{\psi} \), then yields \( \rho_{xx} = \phi_x/2\sqrt{\psi} = \phi_{\rho}/2 \), substituting it into