POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS
FOR SECOND-ORDER SINGULAR NONLINEAR
DIFFERENTIAL EQUATIONS

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Abstract: New existence results are presented for the singular second-order nonlinear boundary value problems \( u'' + g(t)f(u) = 0, \ 0 < t < 1, \ au(0) - \beta u'(0) = 0, \ \gamma u(1) + \delta u'(1) = 0 \) under the conditions \( 0 \leq f \leq M_1, \ m_1 < f \leq \infty, \ \) where \( f = \lim_{u \to 0} f(u)/u, \ f = \lim_{u \to \infty} f(u)/u, \ f = \lim_{u \to 0} f(u)/u, \ f = \lim_{u \to \infty} f(u)/u, \ g \) may be singular at \( t = 0 \) and/or \( t = 1 \). The proof uses a fixed point theorem in cone theory.

Key words: second-order singular boundary value problems; positive solutions; cone; fixed point

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Introduction

In this paper, we shall consider the following singular boundary value problems (BVP)

\[
\begin{align*}
\label{eq:1}
u'' + g(t)f(u) &= 0, \quad 0 < t < 1, \\
au(0) - \beta u'(0) &= 0, \\
\gamma u(1) + \delta u'(1) &= 0,
\end{align*}
\]

where \( \alpha, \beta, \gamma, \delta \geq 0, \rho := \beta \gamma + \alpha \delta > 0, \ f \in C([0, \infty), [0, \infty)), \ g \) may be singular at \( t = 0 \) and/or \( t = 1 \). This problem arises naturally in the study of radially symmetric solutions of nonlinear elliptic equations, non-Newton fluid theory, reaction-diffusion theory and the turbulent flow of a gas in a porous medium. Relatively more results on such boundary value problems were obtained when \( g \) is continuous on \([0, 1]\), for example, positive solutions of non-singular BVP(1) were presented in [1]. Recently, the existence of positive solutions of BVP(1) has been studied in [2], when \( g \in C((0, 1), [0, \infty)), \ 0 < \int_0^1 G(s, s)g(s)ds < \infty, \) and \( f \) is superlinear, that is, \( f_0 = \lim_{u \to 0} f(u)/u = 0, \ f_\infty = \lim_{u \to \infty} f(u)/u = \infty \) or \( f \) is sublinear, that is, \( f_0 = \infty \) and \( f_\infty = 0. \)

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In this paper, we study the existence of positive solutions of BVP(1) under the more general limited condition and integral condition, in detail, \( 0 \leq f_0^+ < M_1, m_1 < f_0^- < M_1, m_1 < f_0^- < \infty \) or \( 0 \leq f_0^+ < M_1, m_1 < f_0^- < \infty \) and \( 0 < \int_0^1 G(s,s)g(s)ds < \infty \), where \( m_1, M_1 \) will be defined in Section 1.

The existence results obtained in this paper will actually improve and extend the main results in \([1, 2]\).

1 Preliminaries and Lemmas

Let \( E \) be a real Banach space, \( P \) a cone in \( E \), \( P \) reduces the partial order "\( \preceq \)" in \( E \), that is, \( x \preceq y \Longleftrightarrow y - x \in P \). Suppose \( G(t,s) \) is the green function of the following boundary problem

\[
\begin{cases}
  u'' = 0, & 0 < t < 1, \\
  au(0) - \beta u'(0) = 0, & \gamma u(1) + \delta u'(1) = 0,
\end{cases}
\]

(2)

given by

\[
G(t,s) = \begin{cases}
  \frac{1}{\rho} (\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \leq s \leq t \leq 1, \\
  \frac{1}{\rho} (\gamma + \delta - \gamma s)(\beta + \alpha t), & 0 \leq t \leq s \leq 1.
\end{cases}
\]

It is obvious that

\[
G(t,s) \preceq G(s,s), \quad 0 \leq t, s \leq 1.
\]

(3)

For the sake of convenience, let us list the following some conditions:

(H1) \( f \in C([0, \infty), [0, \infty)) \);

(H2) \( g \in C((0,1), [0, \infty)) \) and \( 0 < \int_0^1 G(s,s)g(s)ds < \infty \).

By virtue of \( \int_0^1 G(s,s)g(s)ds > 0 \) and \( g \in C((0,1), [0, \infty)) \), there exists \( t_0 \in (0,1) \) such that \( g(t_0) > 0 \). Obviously, there exist \( a, b \in [0,1], a < b \) such that \( t_0 \in (a, b) \). Let

\[
K = \{ u \in C[0,1] | u(t) \geq 0, \min_{s \in [a,b]} u(t) \geq M \| u \| \},
\]

where

\[
\| u \| = \sup_{t \in [0,1]} | u(t) |, \quad M = \min \left[ \frac{\delta + (1 - b)\gamma}{\delta + \gamma}, \frac{ab + \beta}{a + \beta} \right].
\]

Clearly, \( K \) is a cone of \( C[0,1] \) and \( 0 < M < 1 \). From (H2), define an operator \( A: C[0,1] \to C[0,1] \) by

\[
Au(t) = \int_0^1 G(t,s)g(s)f(u(s))ds.
\]

By virtue of (3), we have

\[
Au(t) = \int_0^1 G(t,s)g(s)f(u(s))ds \leq \int_0^1 G(s,s)g(s)f(u(s))ds, \quad t \in [0,1].
\]

So,