Nonlinear Lagrangian Theory for Nonconvex Optimization

C. J. GOH and X. Q. YANG

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Abstract. The Lagrangian function in the conventional theory for solving constrained optimization problems is a linear combination of the cost and constraint functions. Typically, the optimality conditions based on linear Lagrangian theory are either necessary or sufficient, but not both unless the underlying cost and constraint functions are also convex.

We propose a somewhat different approach for solving a nonconvex inequality constrained optimization problem based on a nonlinear Lagrangian function. This leads to optimality conditions which are both sufficient and necessary, without any convexity assumption. Subsequently, under appropriate assumptions, the optimality conditions derived from the new nonlinear Lagrangian approach are used to obtain an equivalent root-finding problem. By appropriately defining a dual optimization problem and an alternative dual problem, we show that zero duality gap will hold always regardless of convexity, contrary to the case of linear Lagrangian duality.

Key Words. Inequality constraints, nonlinear Lagrangian, nonconvex optimization, sufficient and necessary conditions, zero duality gap.

1. Introduction

Consider the following inequality constrained optimization problem:

\[
(P_0) \quad \inf_{x \in X} f_0(x),
\]

\[
\text{s.t. } f_i(x) \leq \theta_i, \quad i = 1, 2, \ldots, m,
\]

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2Research Fellow, Department of Mathematics and Statistics, University of Western Australia, Nedlands, Western Australia, Australia.

3Assistant Professor, Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, PRC.
where the functions \( f_i: \mathbb{R}^n \rightarrow \mathbb{R}, i = 0, 1, 2, \ldots, m \), are continuously differentiable but not necessarily convex, and \( X \) is a subset of \( \mathbb{R}^n \). The constraint \( x \in X \) is intended to include simple constraints, such as simple bounds or linear constraints. Without loss of generality, we assume that the parameter vector
\[
\theta = [\theta_1, \ldots, \theta_m] \in \text{int} \mathbb{R}^m.
\]
Define the set of all feasible solutions to be
\[
X_0 = \{x \in X | f_i(x) \leq \theta_i, i = 1, 2, \ldots, m\}.
\]
Let
\[
g(x) = [f_1(x), f_2(x), \ldots, f_m(x)].
\]
The family of perturbed problems associated with problem \( P_0 \) is defined by
\[
(P_y) \quad \inf_{x \in X} f_0(x) \quad \text{s.t.} \quad g(x) \leq y,
\]
where the vector \( y = [y_1, y_2, \ldots, y_m] \) is a perturbation to the parameter vector \( \theta \) of the original problem \( P_0 \). When \( y = \theta \), the perturbed problem reduces to the original problem \( P_0 \). Let the perturbation function \( w: \mathbb{R}^m \rightarrow \mathbb{R} \) be defined by
\[
w(y) = \inf \{f_0(x) | g(x) \leq y, x \in X\}.
\]
Using the conventional notion that \( \inf \emptyset = +\infty \), \( w \) has an effective domain
\[
\text{dom}(w) = \{y | \exists x \in X, \text{s.t.} g(x) \leq y\}.
\]
Clearly, the perturbation function \( w \) is a monotone nonincreasing function of \( y \). Define the epigraph of \( w(y) \) as the set
\[
\text{epi}(w) = \{(y, y_0) | y \in \text{dom}(w), y_0 \geq w(y)\} \subset \mathbb{R}^{m+1}.
\]
The conventional way of tackling this problem theoretically is via the Lagrangian approach; see for example Ref. 1. To distinguish the conventional Lagrangian approach from our proposed approach, we shall call the former as the linear Lagrangian theory, consistently with the fact that the Lagrangian function is a linear combination of the cost and constraint functions,
\[
L(x, \lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i [f_i(x) - \theta_i].
\]