On Some Reliability Applications of Rice’s Formula for the Intensity of Level Crossings

IGOR RYCHLIK

Center for Mathematical Sciences, Lund University, Box 118, S-221 00 Lund, Sweden
E-mail: igor@maths.lth.se

[Received June 3, 1998; Revised and Accepted March 8, 2001]

Abstract. Let X be a stationary process with absolutely continuous sample paths. If \( E[|X(0)|] \) is finite and if the distribution of \( X(0) \) is absolutely continuous, then, for almost all \( u \), the crossing intensity \( \mu(u) \) of the level \( u \) by \( X(t) \) is given by the generalized Rice’s formula \( \mu(u) = E[|X(0)|]/|X(0)| = |\int_{X(0)} f_{X(t)}(u) du| \). The classical Rice’s formula for \( \mu(u) \), which is valid for a fixed level \( u \), \( \mu(u) = \int |\int_{f_{X(t)}}(z, u) du| dz \), holds under more restrictive technical conditions that can be difficult to check in applications. In this paper it is shown that often in practice the weaker form of Rice’s formula (valid for almost all \( u \)) is sufficient. Three engineering problems are discussed; prediction of fatigue life time; computing the average stress at slams and analysis of crest height of sea waves.

Key words. level crossings, Rice’s formula, fatigue, rainfall damage, wave height

1. Introduction

The celebrated Rice formula for the expected number of times a stationary process \( X(t) \), \( t \in [0, 1] \), “crosses” a fixed level \( u \) has found application in various engineering problems, especially in safety analysis of structures interacting with the environment, for example through wind pressure, ocean waves or temperature variations. The safety of a structure may depend on extreme and rare events such as loads which exceed the strength of a component, or on everyday load variability that may cause changes in the properties of the material, e.g. cracking (fatigue) or other types of aging processes. In the first case, the number of rare events that occur in time or in space is often modeled as a Poisson process. Then, the Rice formula is used to compute the intensity of events, and hence gives the parameters in the Poisson model. In the second case, the aging process may depend both on frequencies of some events as well as their magnitudes. A magnitude of an event is called a “mark”. In Sections 2 and 3 several examples of marks will be given.

The conditions for validity of the Rice formula are well known, see Section 2 for a short review. However, there exist physically motivated models, e.g. when \( X \) is the output of some nonlinear system with Gaussian inputs, for which Rice’s formula can be computed but it is difficult to check the sufficient conditions implying that the computed value is equal to the intensity of crossings. On the other hand, it is proved for absolutely differentiable \( X \), such that \( E[|X(0)|] < \infty \), that the Rice formula is valid for almost all \( u \). The result follows from Banach’s theorem given in Section 2.2.
Our main objective is to demonstrate that such a.e. result can be used efficiently in various important engineering applications involving the so called “rate independent” processes. The properties of rate independent operators have been studied in the monograph by Brokate and Sprekels (1996). Here we give only the definition of such operators.

**Definition 1:** Let $C^1_{[0,t_e]}$ be a space of absolutely continuous functions defined on $[0,t_e]$. A function $g \in C^1_{[0,t_e]}$ will be called a time transformation, if $g$ is increasing (not necessarily strictly) and the range of $g$ is $[0,t_e]$. A functional $H$ defined on $C^1_{[0,t_e]}$ will be called rate independent if for all $\omega \in C^1_{[0,t_e]}$ and all time transformations $g$

$$H(\omega) = H(\omega \circ g),$$

(Examples of time invariant functionals are; suprenum $H(\omega) = \sup_{t \in [0,t_e]} \omega(t)$, total variation of $\omega$, number of crossings of a fixed level by $\omega$ or $H(\omega) = \int_0^t |\dot{\omega}(s)|ds$.)

A rate independent functional $H$ defines an operator $\mathcal{H}$ on $C^1_{[0,t_e]}$ in the following way

$$\mathcal{H}[\omega](t) = H(\omega_t),$$

where, for any $\omega \in C^1_{[0,t_e]}$,

$$\omega_t(s) = \begin{cases} \omega(s) & \text{if } s \leq t, \\ \omega(t) & \text{if } s > t. \end{cases} \quad (1)$$

Important application of rate independent functionals is to describe a memory of a material of the experienced loads, which is then employed to model the accumulated fatigue damage. Section 3 is devoted to description of fatigue processes and the role of Rice’s formula to predict the fatigue life.

There are applications when it is not obvious if the a.e. version of Rice’s formula are adequate to apply. One could object that it is of no use to know that the intensity of hitting a level is “maybe” 6.3 per unit time, because the “maybe” arises from the fact that we do not know whether the level belongs to the Lebesgue set of measure zero for which Rice formula fails or not. Obviously, the level can be “special” in some sense (the process changes discontinuously its properties when passing the level). However, if the value of the level is known only to some accuracy (or if it does not matter if one changes it by a negligible amount), then one can use the Rice formula as it is true for all values of $u$. This is illustrated in Section 2.3, where we compute the average stress at slams, defined as the instances when a ship bow reenters the sea level and in Section 2.4, where we discuss an important problem from oceanography; computation of the distribution of wave crest height.