ABSTRACT. The system \( R^{##} \) of “true” relevant arithmetic is got by adding the \( \omega \)-rule “Infer \( \forall x A x \) from \( A0, A1, A2, \ldots \)” to the system \( R^# \) of “relevant Peano arithmetic”. The rule \( \supset E \) (or “gamma”) is admissible for \( R^{##} \). This contrasts with the counterexample to \( \supset E \) for \( R^# \) (Friedman & Meyer, “Whither Relevant Arithmetic”). There is a Way Up part of the proof, which selects an arbitrary non-theorem \( C \) of \( R^{##} \) and which builds by generalizing Henkin and Belnap arguments a prime theory \( T \) which still lacks \( C \). (The key to the Way Up is a Witness Protection Program, using the \( \imath \)-rule.) But \( T \) may be TOO BIG, whence there is a Way Down argument that produces a better theory \( TR \), such that \( R^{##} \subseteq TR \subseteq T \). (The key to the Way Down is a Metavaluation, on which membership in \( T \) is combined with ordinary truth-functional conditions to determine \( TR \).) The result is a theory that is Just Right, whence it never happens that \( A \supset C \) and \( A \) are theorems of \( R^{##} \) but \( C \) is a non-theorem.

The purpose of this note is to keep a promise. In [1] and [2] (and other places) I announced that the rule \( \gamma \) of detachment for material implication is admissible for the system \( R^{##} \). The proof requires a variant of “Belnap’s Lemma” (dubbed the “Pair extension lemma” in [3] and independently due to Gabbay). That’s for the “Way Up” part of the argument. The “Way Down” involves straightforward “metavaluations” technique, of the sort that I have developed at length in [4] and elsewhere.

I. INTRODUCTION

In the first place, here is Ackermann’s rule \( \gamma \).

\[
(\gamma) \quad \text{From } \sim A \lor B \text{ and } A, \text{ infer } B.
\]

On the usual definition of material \( \supset \), this is just the rule of detachment for material implication, which we express by

\[
(\supset E) \quad \text{From } A \supset B \text{ and } A, \text{ infer } B.
\]

Since the latter will be more familiar to readers who are not relevant cognoscenti, I will henceforth refer to \( \gamma \) as \( \supset E \) (mostly).

It is interesting (and much discussed) that the question of the admissibility of \( \supset E \) arises at all in relevant logics. Most logics and their accompanying theories get it for free, either because it is already a primitive rule (often the only one) or because it is at least backed up by the...
theorem

(1) \((A \supset B) \& A \rightarrow B.\)

In fact, \(\supset E\) started out as a primitive rule of Ackermann’s system II’ of strenge Implikation, whence it has been argued (in effect, by Maksimova) that closure under \(\supset E\) is built into the notion of II’-deducibility.

But II’ lacks the theorem scheme (1). This sufficiently disturbed Anderson and Belnap that, when they reformulated strenge Implication as their system \(E\) of entailment in the 1950’s, they chopped \(\supset E\) from among the primitive rules. This left a question: Does \(E\) have the same stock of theorems of II’? Other chopping and changing having been (relatively) trivial, an affirmative answer hung for a while on the admissibility of \(\supset E\) for \(E\). And this was, for a while, one of the major open problems for that system.3

It was pleasant to find, by the arguments recounted in [3], that not only was \(\supset E\) admissible for logics like \(E\) and \(R\); but it remained admissible for the first-order extensions \(E^\forall x\), \(R^\forall x\), etc. of these sentential logics.4 For we are interested in \(\supset E\) not merely for logics but also for the theories that respect these logics. That is, whenever we formulate a concrete theory using a relevant logic, we want to know whether the set of theorems of that theory is closed under \(\supset E\).

II. FORMULATION OF R# AND R##

Let \(L\) be a Logic (or other Background Theory), fixed in context. We write

\[ (D \leq) \ A \leq B \text{ iff } \vdash_L A \rightarrow B. \]

That is, \(\leq\) is, by the lights of \(L\), provable entailment. As in [10] (and in related though independent work by Fine), I call a set \(T\) of formulas in the vocabulary of \(L\) an \(L\)-theory provided that \(T\) is closed under conjunction and provable \(L\)-entailment. That is, \(T\) is an \(L\)-theory provided that, for all \(A, B\) in its vocabulary, we have

\[ (\leq E) \text{ If } A \leq B \text{ then if } A \in T \text{ then } B \in T, \text{ and} \]

\[ (&I) \text{ If } A \in T \text{ and } B \in T \text{ then } A \& B \in T. \]

Note that, as [3] also points out, an \(L\)-theory is not necessarily required to respect its own modus ponens.5 It is required to respect entailment according to its Background Theory \(L\).