Stationary Generalization of the Bonnor Magnetic Dipole Solution

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An exact asymptotically flat 3-parameter solution of the Einstein–Maxwell equations is presented that reduces to the Bonnor magnetic dipole solution in the magnetostatic limit, and to the Tomimatsu–Sato $\delta = 2$ solution in the stationary pure vacuum limit. This solution is the simplest possible one admitting the polynomial representation in the spheroidal coordinates $(x, y)$ and able to describe the exterior field of a magnetized spinning mass. A multipole criterion for the choice of the parameters in the Einstein–Maxwell spacetimes is also formulated.

**KEY WORDS**: Einstein–Maxwell equations; magnetic dipole; asymptotically flat solution

1. INTRODUCTION

Bonnor was the first to obtain an exact asymptotically flat 2-parameter solution of the Einstein–Maxwell equations appropiate for the description of the exterior field of a massive magnetic dipole [1] by applying his theorem [2] to the Kerr metric [3], and since then it has been a long standing problem to obtain a stationary generalization of this solution. Bonnor’s magnetostatic metric was later generalized by Kramer and Neugebauer [4] to include an additional parameter of charge, the resulting metric being already a stationary one, but without a stationary pure vacuum limit.

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Since the problem of introducing an independent parameter of angular momentum into the Bonnor metric is equivalent to finding a magnetic generalization of the Tomimatsu–Sato $\delta = 2$ solution [5], it is worth mentioning that the search for such an electrovac solution was pioneered by Kinnersley who outlined a procedure for constructing a nine-parameter stationary electrovac metric possessing a magnetic dipole parameter [6]. Kinnersley’s idea was later realized in [7], where a rational function solution was obtained that could be interpreted in its special case as a magnetized superextreme Tomimatsu–Sato $\delta = 2$ solution. The fact that the latter solution was not applicable to the more interesting subextreme case possibly explains why the authors of [7] restricted their consideration to the derivation of the Ernst complex potentials [8] defining the solution, not even being fully confident in the physical interpretation of the parameters associated with the electromagnetic field.

The aim of our paper is to present a stationary generalization of the Bonnor metric in a simple form that would be equally applicable both to the sub- and superextreme cases of the magnetized spinning sources.

2. THE ERNST COMPLEX POTENTIALS AND METRIC FUNCTIONS

The reported solution has been obtained with the aid of Sibgatullin’s integral method [9] applied to the axis data of the form

\[
\begin{align*}
\zeta(\rho = 0, z) &\equiv e(z) = \frac{z^2 - 2(m + ia)z + m^2 - a^2 - \epsilon^2}{z^2 + 2(m - ia)z + m^2 - a^2 - \epsilon^2}, \\
\Phi(\rho = 0, z) &\equiv f(z) = \frac{2ic' z^2 + 2(m - ia)z + m^2 - a^2 - \epsilon^2}{z^2 + 2(m - ia)z + m^2 - a^2 - \epsilon^2}, \\
c^2 &\equiv \frac{\epsilon^2}{m^2 - a^2},
\end{align*}
\]

(1)

where $\zeta$ and $\Phi$ stand for the Ernst complex potentials, $\rho$ and $z$ are the Weyl–Papapetrou cylindrical coordinates, and $m$, $a$, $c'$ are arbitrary real constants associated respectively with the total mass, angular momentum and magnetic dipole moment of the source. When $a = 0$, one obtains from (1) the axis data of the Bonnor solution; on the other hand, with $c' = 0$, one recovers from (1) the potential $e(z)$ of the Tomimatsu–Sato $\delta = 2$ solution representing both the sub- and superextreme cases. The parameters in (1) are chosen in such a way that the algebraic equation

\[
e(z) + \bar{e}(z) + 2f(z)\bar{f}(z) = 0
\]

(2)