Gravity and Signature Change

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The use of proper "time" to describe classical "spacetimes" which contain both Euclidean and Lorentzian regions permits the introduction of smooth (generalized) orthonormal frames. This remarkable fact permits one to describe both a variational treatment of Einstein's equations and distribution theory using straightforward generalizations of the standard treatments for constant signature.

KEY WORDS: Signature change; smooth orthonormal frames

1. INTRODUCTION

A signature-changing spacetime is a manifold which contains both Euclidean and Lorentzian regions. Signature-changing metrics must be either degenerate (vanishing determinant) or discontinuous, but Einstein's equations implicitly assume that the metric is nondegenerate and at least...
continuous.\footnote{This can be weakened \cite{[1]} to allow locally integrable metrics admitting a square-integrable weak derivative. Discontinuous metrics do not satisfy this condition.} Thus, in the presence of signature change, it is not obvious what “the” field equations should be.

For discontinuous signature-changing metrics, one can derive such equations from a suitable variational principle \cite{[2]}. This turns out to follow from the existence in this case of a natural generalization of the notion of orthonormal frame. The standard theory of tensor distributions, as well as the usual variation of the Einstein–Hilbert action, can both be expressed in terms of orthonormal frames, and thus generalize in a straightforward manner to these models. No such derivation is known for continuous signature-changing metrics. Our key point is that although signature change requires the metric to exhibit some sort of degeneracy, there is in the discontinuous case a more fundamental field, namely the (generalized) orthonormal frame, which remains smooth.

We introduce here two simple examples in order to establish our terminology. A typical continuous signature-changing metric is

\[ ds^2 = t \, dt^2 + a(t)^2 \, dx^2 \]  

(1)

whereas a typical discontinuous signature-changing metric is

\[ ds^2 = \text{sgn}(\tau) \, d\tau^2 + a(\tau)^2 \, dx^2. \]  

(2)

Away from the surface of signature change at \( \Sigma = \{ t = 0 \} = \{ \tau = 0 \} \), these metrics are related by a smooth coordinate transformation, with \( \tau \) denoting proper “time” away from \( \Sigma \). However, since \( d\tau = \sqrt{|t|} \, dt \), the notions of smooth tensors associated with these coordinates are different at \( \Sigma \), corresponding to different differentiable structures.

We argue here in favor of the discontinuous metric approach, both physically and mathematically; physically, because of the fundamental role played by proper time, and mathematically, because of the geometric invariance of the unit normal to the surface of signature change. The resulting (generalized) orthonormal frames provide a clear path leading to a straightforward generalization of both Einstein’s equations and the theory of tensor distributions.

2. PHYSICS

A standard tool in the description of physical processes is the introduction of an orthonormal frame. Physical quantities can be expressed...