NUMERICAL METHODS.
FINITE-DIFFERENCE EQUATIONS

Periodic Solutions of a System
of Linear Difference Equations
with Continuous Argument

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Received July 15, 1999

1. INTRODUCTION
Consider the system of linear difference equations

\[ x(t) = Ax(t - 1) + f(t) \] (1.1)

with continuous argument \( t \in \mathbb{R} \), where \( x \in \mathbb{R}^n \), \( A \) is a real constant \( n \times n \) matrix, and \( f(t) \) is a \( T \)-periodic vector function of dimension \( n \). There is a vast literature in which the theory of such systems is developed; a rather comprehensive bibliography can be found, e.g., in [1, 2]. Most papers deal with the construction of a general or special solution of such systems [4–7]; in particular, the general form and conditions for the existence and uniqueness of periodic solutions of finite-difference systems with continuous argument in the case of an integer period were obtained, for example, in [2, 5, 6]. The existence of periodic solutions of difference equations was analyzed in [3, Chap. 5], where the considerations were not restricted to the case of an integer period. However, the assumption on the Fredholm property of the operator of the system restricted the class of periodic solutions to a finite-dimensional space, which is solely due to the method based on the Fredholm alternative.

The main goal of the present paper is to clarify conditions under which system (1.1) has a solution periodic with period rationally commensurable with the time increment; the space of such solutions can be infinite-dimensional. In Section 2, we consider the problem on the existence of a periodic solution (with rational or irrational period) of the homogeneous system corresponding to (1.1). The existence conditions are given in Theorem 1. The form of such a solution (if it exists at all) is discussed in Section 3 and given in Theorem 2. Section 4 contains the main results, stated in Theorems 3 and 4. It deals with existence conditions for periodic solutions of the nonhomogeneous system. If these conditions are satisfied, then one can obtain a periodic solution of system (1.1) in the form described in Theorems 5 and 6 (which are proved in Section 5). In conclusion, we consider examples.

2. THE EXISTENCE OF PERIODIC SOLUTIONS
OF THE HOMOGENEOUS SYSTEM
Consider the homogeneous system

\[ x(t) = Ax(t - 1) \] (2.1)

corresponding to (1.1). There exists a nondegenerate transformation \( y = Sx \) reducing (2.1) to the system

\[ y(t) = Jy(t - 1), \] (2.2)

where \( J \) is a Jordan normal form of the matrix \( A \). System (2.2) is defined over the field of complex numbers.

We relate the investigation of system (2.2) to the analysis of the scalar equation

\[ y(t) = gy(t - 1) \] (2.3)
with a complex constant coefficient \( g \). If \( g = 0 \), then Eq. (2.3) has only the trivial solution \( y(t) = 0, t \in \mathbb{R} \); but if \( g \neq 0 \), then the general solution of Eq. (2.3) is given by the formula

\[
y(t) = g^t c(t), \quad t \in \mathbb{R},
\]

where \( c(t) \) is an arbitrary 1-periodic complex-valued function.

Let us prove the following auxiliary assertions.

**Lemma 1.** Let \( |g| \neq 1 \). Then Eq. (2.3) does not have nontrivial periodic solutions.

**Proof.** If \( g \neq 0 \), \( |g| \neq 1 \), and \( c(t) \) does not identically vanish for \( t \in [0, 1] \), then the solutions (2.4) of Eq. (2.3) either asymptotically tend to zero as \( t \) grows or are unbounded on the time axis.

**Lemma 2.** Let \( |g| = 1 \), and let \( \arg g/(2\pi) \) be an irrational number. Then the set of periodic solutions of Eq. (2.3) is given by the formula

\[
y(t) = e^{i\varphi t} c(t), \quad \text{where } i = \sqrt{-1}, \ \varphi = \arg g, \ -\pi < \varphi \leq \pi,
\]

and \( c \) is an arbitrary complex constant.

**Proof.** The period of the function \( g^t = e^{i\varphi t} \) is rationally incommensurable with any number of the form \( 1/N \) with integer \( N \). Therefore, the solution (2.4) of Eq. (2.3) can have only an irrational period, which is possible if and only if \( c(t) = c = \text{const} \).

**Lemma 3.** Let \( |g| = 1 \), and let \( \arg g/(2\pi) \) be a rational number \( \varphi \) [either \( \arg g/(2\pi) = 0 \), or \( \arg g/(2\pi) = k/m \), where \( k \) and \( m \) are positive integers and \( k/m \) is an irreducible fraction]. The set of periodic solutions of Eq. (2.3) is given by the formula

\[
y(t) = e^{i\varphi t} c(t) \quad (i = \sqrt{-1}),
\]

where \( c(t) \) is an arbitrary 1-periodic function for the set of \( m \)-periodic solutions and \( c(t) \) is an arbitrary \( 1/|k| \)-periodic function for the set of \( m/|k| \)-periodic solutions (here \( \varphi = 2\pi k/m, -\pi < \varphi \leq \pi \)). If \( \varphi = 0 \), then \( c(t) \) is an arbitrary 1-periodic function for the set of 1-periodic solutions, \( c(t) \) is an arbitrary \( 1/N \)-periodic function \( (N \) is a positive integer) for \( 1/N \)-periodic solutions, and \( c(t) = c \) is an arbitrary constant for the set of constant solutions of Eq. (2.3).

**Proof.** If \( \varphi \neq 0 \), then the solution (2.4) is not constant for any \( c(t) \) other than identical zero. If \( \varphi = 0 \), then, for the solution of Eq. (2.3) to be constant, it is necessary and sufficient that \( c(t) = c = \text{const} \) in (2.4). If \( \varphi = 0 \) and \( N \) is an arbitrary positive integer, then, for the solution of Eq. (2.3) to be \( 1/N \)-periodic, it is necessary and sufficient that \( c(t) \) is an \( 1/N \)-periodic function. In particular, if \( N = 1 \), then the solution of Eq. (2.3) is 1-periodic. If \( \varphi = 2\pi k/m \neq 0 \) \( (-\pi < \varphi \leq \pi) \), then the least period of the function \( g^t = e^{i\varphi t} \) is \( m/|k| \). For the solution of Eq. (2.3) given by (2.4) to be \( m/|k| \)-periodic, it is necessary and sufficient that \( c(t) \) is an \( 1/|k| \)-periodic function. If we do not impose additional constraints on the set of 1-periodic functions \( c(t) \) occurring in (2.4), then this formula defines the set of \( m \)-periodic solutions of Eq. (2.3).

**Theorem 1.** System (2.1) has periodic solutions if and only if there exist eigenvalues of the matrix \( A \) with absolute values equal to 1.

**Proof.** System (2.1) has periodic solutions if and only if system (2.2) has such solutions. If all eigenvalues of the matrix \( A \) correspond only to simple elementary divisors, then system (2.2) splits into \( n \) equations of the form (2.3), and the validity of the theorem follows from Lemmas 1–3. If there is at least one multiple eigenvalue \( \varrho \) corresponding to \( r \) nonsimple elementary divisors \( (r > 1) \), then the matrix \( J \) has the corresponding block of order \( r \) inducing the system

\[
\begin{align*}
y_1(t) &= \varrho y_1(t - 1) + y_2(t - 1), \\
y_2(t) &= \varrho y_2(t - 1) + y_3(t - 1), \\
&\vdots \\
y_{r-1}(t) &= \varrho y_{r-1}(t - 1) + y_r(t - 1), \\
y_r(t) &= \varrho y_r(t - 1).
\end{align*}
\]

Differential Equations Vol. 37 No. 4 2001