MULTIPLICATION MODULES AND A THEOREM OF P. F. SMITH

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[Communicated by: László Fuchs]

Abstract

P. F. Smith [7, Theorem 8] gave sufficient conditions on a finite set of modules for their sum and intersection to be multiplication modules. We give sufficient conditions on an arbitrary set of multiplication modules for the intersection to be a multiplication module. We generalize Smith’s theorem, and we prove conditions on sums and intersections of sets of modules sufficient for them to be multiplication modules.

0. Introduction

Let $R$ be a commutative ring and $M$ an $R$-module. Let $K$ and $L$ be submodules of $M$. The residual of $K$ by $L$, denoted $[K : L]$, is the set of all elements $x$ in $R$ such that $xL \subseteq K$. The annihilator of an $R$-module $M$ denoted by $\text{ann}(M)$ is $[O : M]$ and for each $m \in M$, the annihilator of $m$, denoted by $\text{ann}(m)$ is $[O : Rm]$. An $R$-module $M$ is a multiplication module if for every $R$-submodule $K$ of $M$ there exists an ideal $I$ of $R$ such that $K = IM$, [3]. Equivalently, $K = [K : M]M$. It is clear that every cyclic module is a multiplication module. Let $P$ be a maximal ideal of $R$. An $R$-module $M$ is called $P$-torsion if for each $m \in M$ there exists $p \in P$ such that $(1 - p)m = 0$. On the other hand $M$ is called $P$-cyclic if there exist $x \in M$ and $q \in P$ such that $(1 - q)M \subseteq xR$. El-Bast and Smith [4] showed that an $R$-module $M$ is multiplication if and only if for every maximal ideal $P$ of $R$ either $M$ is $P$-torsion or $P$-cyclic. In [2] it is mentioned that an $R$-module $M$ is $P$-torsion if and only if $MP = 0_P$. It follows that $M$ is not $P$-torsion if and only if $\text{ann}(m) \subseteq P$ for some $m \in M$. In Section 1 (Theorem 3) we give a sufficient condition for the intersection of a set of multiplication submodules to be a multiplication module. In Section 2, Proposition 9 and Theorem 11 give a generalization of Smith’s theorem, and Theorem 12 shows necessary conditions on sums and intersections of modules for them to be multiplication modules. All rings are commutative with identity, and all modules are unitary. For the basic concepts used in this paper, consult [5] and [6].

Mathematics subject classification number: 13C13, 13A15

Key words and phrases: multiplication module, multiplication ideal.
1. Intersection of multiplication modules

We start with the following lemma of P. F. Smith [7] which characterizes multiplication modules.

**Lemma 1.** Let $R$ be a ring and $M$ an $R$-module. A submodule $K$ of $M$ is multiplication if and only if for each maximal ideal $P$ of $R$ either $K$ is $P$-torsion or there exist a multiplication submodule $L$ of $K$ and $p \in P$ such that $(1 - p)K \subseteq L$.

**Corollary 2.** Let $R$ be a ring and $M$ an $R$-module. A submodule $K$ of $M$ is multiplication if and only if for each maximal ideal $P$ of $R$ either $K$ is $P$-torsion or there exist a multiplication submodule $L$ of $M$ containing $K$ and $p \in P$ such that $(1 - p)L \subseteq K$.

**Proof.** The necessity is immediate by the above lemma and letting $K = L$. Conversely, suppose that for each maximal ideal $P$ of $R$ either $K$ is $P$-torsion or there exist a multiplication submodule $L$ of $M$ containing $K$ and $p \in P$ such that $(1 - p)L \subseteq K$. Let $Q$ be a maximal ideal of $R$. Suppose that $K$ is not $Q$-torsion. Then there exist a multiplication submodule $L$ of $M$ containing $K$ and $q \in Q$ such that $(1 - q)L \subseteq K$. As $K$ is not $Q$-torsion, it follows that $L$ is not $Q$-torsion and hence by Lemma 1 there exist a multiplication submodule $N$ of $L$ and $q' \in Q$ such that $(1 - q')L \subseteq N$. In this case

$$(1 - q)(1 - q')K \subseteq (1 - q)(1 - q')L \subseteq (1 - q)L$$

As $(1 - q)L$ is a multiplication submodule of $K$ [4, Corollary 1.4], we infer from Lemma 1 that $K$ is a multiplication module.

The next result gives a sufficient condition for the intersection of a collection (not necessarily finite) of multiplication submodules of an $R$-module to be a multiplication module.

**Theorem 3.** Let $R$ be a ring and $N_{\lambda}$ ($\lambda \in \Lambda$) a collection of multiplication submodules of an $R$-module $M$, and let $N = \bigcap_{\lambda \in \Lambda} N_{\lambda}$. Then the following statements are equivalent:

(i) $N$ is multiplication and $IN = \bigcap_{\lambda \in \Lambda} IN_{\lambda}$ for every ideal $I$ of $R$.

(ii) $\sum_{\lambda \in \Lambda} [N : N_{\lambda}] + \text{ann}(n) = R$ for all $n \in N$.

**Proof.** Suppose (i) is satisfied. Let $n \in N$. Suppose that

$$\sum_{\lambda \in \Lambda} [N : N_{\lambda}] + \text{ann}(n) \neq R.$$