THE DIOPHANTINE EQUATION

\[(a^n - 1)(b^n - 1) = x^2\]

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Abstract

Let \(a\) and \(b\) given unequal positive integers; it is desired to determine the positive integer solutions \(n\) and \(x\) of the equation of the title. Some special cases have recently been considered, and here some general results and conjectures are presented.

0. Introduction

Clearly \(a\) and \(b\) are interchangeable; we shall usually only state results in one direction, omitting the phrase “or vice-versa” throughout. The problem for general \(a\) and \(b\) presents two facets, a “bright side” which enables many special values of \(\{a, b\}\) to be solved, and a much more difficult aspect in which a number of eminently reasonable conjectures may be stated, but whose proofs appear to be rather difficult. In the hope that these conjectures may facilitate further research, they are included in the second section.

1. The bright side

Szalay [6] has shown that there is no solution if \(\{a, b\} = \{2, 3\}\), only the solution \(n = 1\) for \(\{2, 5\}\), and for \(\{2, 2^k\}\) there is no solution with \(k \geq 2\) except for \(n = 3\) and \(k = 2\). Hajdu & Szalay [4] have shown that there is no solution for \(\{2, 6\}\) and that for \(\{a, a^k\}\) there are no solutions with \(k \geq 2\) and \(kn > 2\) except for the three cases \([a, n, k] = [2, 3, 2], [3, 1, 5]\) and \([7, 1, 4]\). It is the object of this note to generalise these results.

**Result 1.** If \(a^l = b^k\), then there are only the following solutions:
(a) for any \(c \geq 2\), \(a = c^2 - 1\); \(b = a^2\) with \(n = 1\);

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Proof. Without loss of generality we assume that $l > k \geq 1$ and then for some integer $g$, $a = g^c$ and $b = g^\lambda$ where $\lambda = l/(k,l)$ and $\kappa = k/(k,l)$ are coprime and then our equation becomes

$$\left(\frac{g^{\lambda n} - 1}{g^n - 1}\right) \left(\frac{g^\kappa n - 1}{g^n - 1}\right) = \left(\frac{x}{g^n - 1}\right)^2$$

where $\lambda \geq 2$ and the factors on the left hand side are coprime integers. Hence each must be a square. Then $\left(\frac{a^{\lambda n} - 1}{g^n - 1}\right) = x_1^2$. If $\lambda > 2$, then by a result of Ljunggren [5] this is possible only for the two cases $\lambda = 4, g^n = 7$ and $\lambda = 5, g^n = 3$, the last two solutions.

If $\lambda = 2$, then $\kappa = 1$ and we have $g^n + 1 = c^2$ which is known [1] to have no solutions in positive integer if $n \geq 1$ with the exception of $2^4 + 1 = 3^2$. The exception gives the second solution, and otherwise we have $n = 4, a = 7, b = 1$. In what follows, let $(a^n - 1, b^n - 1) = D$; then $a^n - 1 = Dy^2, b^n - 1 = Dz^2$ and $(y, z) = 1$.

Result 2. There is no solution with $4 | n$, except for $(a, b) = \{13, 239\}$ with $n = 4$.

Proof. If $4 | n$, then the equation $v^4 = Du^2 + 1$ has two distinct solutions in positive integers $v$ and $u$. As has been shown in [2], this can occur only with $D = 1785, 7140$ and 28560 all giving $v = 13$ and 239.

There are obviously many cases in which solutions are possible for $n = 1$ and 2, and we deal with these easy cases next.

Result 3. The case $n = 1$ occurs if and only if for some positive integers $\lambda\mu\upsilon$ $a = 1 + \lambda\mu^2, b = 1 + \lambda\upsilon^2$.

Proof. Trivial.

Result 4. The case $n = 2$ occurs if and only if $a = v_r(c), b = v_s(c)$ for some positive integers $c, r$ and $s$, where $c^2 - 1 = (a^2 - 1, b^2 - 1)$ and $(r, s) = 1$, and the polynomials $\{v_n(c)\}$ are defined by $v_0(c) = 1; v_1(c) = c; v_n+2(c) = 2cv_{n+1}(c) - v_n(c), n \geq 0$.

Proof. We have $a^2 - 1 = Dy^2, b^2 - 1 = Dz^2$ where $(y, z) = 1$, and if the fundamental solution of $v^2 - Du^2 = 1$ is $c + d\sqrt{D}$, then $a = v_r(c), y = d\upsilon_r(c), b = v_s(c), z = d\upsilon_s(c)$ where the polynomials $\{u_n(c)\}$ satisfy $u_0(c) = 0; u_1(c) = 1$ and the same recurrence relation as $\{v_n(c)\}$. Since $(y, z) = 1$ it follows that we must