ON THE LIMIT DISTRIBUTION OF INTEGRALS OF SHOT-NOISE PROCESSES

A. B. Il’enko

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We establish limit theorems for integrals of shot-noise processes and study the asymptotic behavior of the moments of integrals of this type.

1. Introduction

Let \((\zeta(s), s \in \mathbb{R})\) be a Lévy process, i.e., a stochastically-continuous homogeneous random process with independent increments and let \(\zeta(0) = 0\). Without loss of generality, we can assume that this process is centered and does not have a Gauss component. In view of the homogeneity of the process, its Lévy measure has the form

\[
\Pi(ds, dx) = ds \times \Pi(dx),
\]

where \(\Pi(dx)\) is the \(\sigma\)-finite Borel measure on \(\mathbb{R}\). For positive integers \(p\), we introduce the spectral moments

\[
\Pi_p = \int_{-\infty}^{\infty} x^p \Pi(dx), \quad \Pi_p = \int_{-\infty}^{\infty} |x|^p \Pi(dx)
\]

and assume in what follows that \(\Pi_2 < \infty\).

For every function \(g \in L_2(\mathbb{R})\), the mean-square integral

\[
\theta(t) = \int_{-\infty}^{\infty} g(t-s) d\zeta(s), \quad t \in \mathbb{R},
\]

defines a stationary shot-noise process (see, e.g., [1]). We call the function \(g\) a response function and assume that it belongs to both the space \(L_2(\mathbb{R})\) and the space \(L_1(\mathbb{R})\).

Denote

\[
\Theta(T) = \int_0^T \theta(u) du, \quad T > 0.
\]

In the present paper, we study the asymptotic behavior of the integrals \(\Theta(T)\), which are regarded as Riemann \(L_2(\Omega)\)-integrals. In Sec. 1, we investigate necessary and sufficient conditions for the function \(g\) under which the random variables \(\Theta(T)\) are uniformly \(L_2(\Omega)\)-bounded \([\text{and} \ L_p(\Omega)\)-bounded\]. Limit distributions are
studied in Sec. 2. In Sec. 3, we investigate the relationship between the growth rate of the moments \( \Theta(T) \) and the properties of the function \( g \). In the present paper, we continue our investigations begun in [2].

**1. Conditions for Uniform \( L_2(\Omega) \)-Boundedness of Integrals \( \Theta(T) \)**

The characteristic function of the random variable \( \Theta(T) \) has the form

\[
\phi_T(\lambda) = \exp \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \exp \left( i\lambda x \int_{s}^{s+T} g(u) du \right) - 1 - i\lambda x \int_{s}^{s+T} g(u) du \right] ds \Pi(dx), \quad \lambda \in \mathbb{R},
\]

and its cumulants are given by the relation

\[
k_p(\Theta(T)) = \Pi_p \int_{-\infty}^{\infty} \left[ \int_{s}^{s+T} g(u) du \right]^p ds, \quad p \in \mathbb{N}.
\]

The statement below gives necessary and sufficient conditions for the response function \( g \) under which the integrals \( \Theta(T) \) are uniformly \( L_2(\Omega) \)-bounded with respect to \( T > 0 \).

**Theorem 1.** In order that the integrals \( \Theta(T), \ T > 0, \) have the second moment bounded with respect to \( T, \) it is necessary and sufficient that the following conditions be satisfied:

1. \( \int_{-\infty}^{\infty} g(u) du = 0; \)

2. \( \sup_{A \geq 0} \int_{0}^{\infty} g(u) g(v) \min \{u, v, A\} du dv < \infty; \)

3. \( \sup_{A \geq 0} \int_{0}^{\infty} g(-u) g(-v) \min \{u, v, A\} du dv < \infty. \)

**Proof.** It follows from relation (3) that the condition

\[
\sup_{T > 0} \mathbb{E} \Theta^2(T) < \infty
\]

is equivalent to the condition

\[
\sup_{T > 0} \int_{-\infty}^{\infty} \left[ \int_{s}^{s+T} g(u) du \right]^2 ds < \infty.
\]