Acta Applicandae Mathematicae **73:** 175–219, 2002.

Universal Central Extensions of Lie Groups

KARL-HERMANN NEEB
University of Technology Darmstadt, Schlossgartenstrasse 7, D-64289 Darmstadt, Germany.
e-mail: neeb@mathematik.tu-darmstadt.de

(Received: 12 June 2001)

Abstract. We call a central Z-extension of a group G weakly universal for an Abelian group A if the correspondence assigning to a homomorphism $Z \to A$ the corresponding A-extension yields a bijection of extension classes. The main problem discussed in this paper is the existence of central Lie group extensions of a connected Lie group G which is weakly universal for all Abelian Lie groups whose identity components are quotients of vector spaces by discrete subgroups. We call these Abelian groups regular. In the first part of the paper we deal with the corresponding question in the context of topological, Fréchet, and Banach–Lie algebras, and in the second part we turn to the groups. Here we start with a discussion of the weak universality for discrete Abelian groups and then turn to regular Lie groups A. The main results are a Recognition and a Characterization Theorem for weakly universal central extensions.

Mathematics Subject Classifications (2000): 22E65, 17E65.

Key words: central extension, Lie algebra, infinite-dimensional Lie group, universal central extension, Lie algebra cohomology, period map.

Introduction

If G is a perfect group, then there exists a universal central extension $q\colon \widehat{G} \to G$ which has the property that for any other central extension $q_1\colon \widehat{G}_1 \to G$ there exists a unique homomorphism $\phi\colon \widehat{G} \to \widehat{G}_1$ with $q_1 \circ \phi = q$. The kernel of q is $H_2(G)$, the second homology group of G ([19], [16], p. 227).

Similar results hold for Lie algebras. For every perfect Lie algebra $\mathfrak{g}$ there exists a universal central extension $q\colon \hat{\mathfrak{g}} \to \mathfrak{g}$ such that for any other central extension $q_1\colon \hat{\mathfrak{g}}_1 \to \mathfrak{g}$ there exists a unique Lie algebra homomorphism $\phi\colon \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}_1$ with $q_1 \circ \phi = q$. Here the kernel can be identified with the second Lie algebra homology space $H_2(\mathfrak{g})$ ([19], [16], p. 228).

The main purpose of this paper is to understand under which circumstances similar results hold for Lie groups. Here we work with not necessarily finite-dimensional Lie groups which are modeled over sequentially complete locally convex spaces ([8]) and consider only those central extensions $q\colon \widehat{G} \to G$ which are locally trivial smooth principal bundles, i.e. which admit smooth local sections. Moreover, we restrict the class of kernels to those Abelian Lie groups Z which are regular in the sense that their identity component is the quotient of a vector space

by a discrete subgroup. All known Abelian Lie groups are regular. Both restrictions are vacuous for finite-dimensional groups, and the second for Banach–Lie groups.

Our main tool to address central extensions in this context are the results of [10] relating them to central extensions of the corresponding Lie algebras. This is why the first three sections of the paper are devoted to (universal) central extensions $q\colon \hat{\mathfrak{g}} \to \mathfrak{g}$ of topological Lie algebras which are linearly split in the sense that they have a continuous linear section (which of course does not have to be a Lie algebra homomorphism). This assumption is crucial because otherwise it would be impossible to parameterize the equivalence classes by objects that one could calculate for specific Lie algebras since extension classes of topological vector spaces would enter the picture, and the groups formed by these extension classes seem to be quite inaccessible.

In Section 1 we discuss central extensions of topological Lie algebras in general. Here a crucial result is an exact sequence

$$\begin{aligned} \mathbf{0} &\longrightarrow \operatorname{Hom}(\mathfrak{g}, \mathfrak{a}) \longrightarrow \operatorname{Hom}(\hat{\mathfrak{g}}, \mathfrak{a}) \\ &\longrightarrow \operatorname{Lin}(\mathfrak{z}, \mathfrak{a}) \xrightarrow{\delta_{\mathfrak{a}}} H^2_c(\mathfrak{g}, \mathfrak{a}) \longrightarrow H^2_c(\hat{\mathfrak{g}}, \mathfrak{z}, \mathfrak{a}) \to \mathbf{0} \end{aligned} \tag{0.1}$$

associated to a central extension $\mathfrak{z} \hookrightarrow \hat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ and a topological vector space $\mathfrak{a}$, where H^2_c denotes the continuous Lie algebra cohomology, Hom stands for continuous Lie algebra homomorphisms, and Lin for continuous linear maps. We call the central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathfrak{z}$ weakly universal for $\mathfrak{a}$ if the homomorphism $\delta_{\mathfrak{a}}$ in (0.1) is bijective. This concept is weaker than the universality used in the algebraic context and makes it possible to discuss universality properties for restricted classes of spaces $\mathfrak{a}$. This turns out to be a good strategy to split the problem into tractable pieces. We will see in particular that for each finite-dimensional Lie algebra $\mathfrak{g}$ all difficulties vanish and that there exists a unique central extension which is weakly universal for all spaces $\mathfrak{a}$. This extension is universal in the sense defined above if and only if the Lie algebra $\mathfrak{g}$ is perfect.

In Section 1 we also discuss uniqueness properties for other classes of infinite-dimensional Lie algebras, but the hard part is to decide when weakly universal central extensions exist. This question is discussed in Section 2 for Fréchet–Lie algebras. The restriction to this class of Lie algebras is natural because on the one hand, it is natural to restrict to locally convex spaces in order to have natural topologies on tensor products, and on the other hand, it is very helpful to have the Open Mapping Theorem. The main result of Section 2 is an existence criterion for a central extension which is weakly universal for all complete locally convex spaces. Our criterion is always satisfied if $\mathfrak{g}$ is (algebraically) perfect and its second cohomology space is finite-dimensional. In the short Section 3 we briefly discuss certain refinements for the class of Banach–Lie algebras.

The structure of Sections 4 and 5 is similar, but here we work on the group side. Section 4 is parallel to Section 1. Here we derive for a central Lie group extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ and each Abelian Lie group A an exact sequence

$$\mathbf{1} \to \operatorname{Hom}(G, A) \longrightarrow \operatorname{Hom}(\widehat{G}, A) \longrightarrow \operatorname{Hom}(Z, A)$$