STABILITY IN THE C-NORM AND W^1_∞-NORM OF CLASSES OF LIPSCHITZ FUNCTIONS OF ONE VARIABLE

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Abstract: In the framework of Kopylov's ω-stability concept, we study some stable classes of Lipschitz functions of one real variable. We obtain an exhaustive (nontrivial) classification for these classes and establish the relevant stability estimates in the W^1_∞-norm.

Keywords: stability, classes of Lipschitz functions of one variable

Introduction

Stability of a mapping class ℘ means on a heuristic level that local proximity of a mapping f to mappings in ℘ implies global proximity of f to them. We illustrate the kind of objects that are studied in this article by the following example. Let G be a compact subset of R. Consider the class ℘ = {g : Δ → R | g is convex and g′ ∈ G almost everywhere} of convex mappings on the various intervals Δ ⊂ R. If the interior int G is nonempty then the class ℘ is unstable in a sense. Indeed, if int G = ∅ then there is a nonconvex everywhere differentiable function f : (0, 1) → R such that f′(x) ∈ G for every x ∈ (0, 1). By the definition of derivative, we can approximate f locally (in a small neighborhood of each point x ∈ (0, 1)) with arbitrary accuracy by mappings in ℘. However, in the global sense, f is separated over the whole interval (0, 1) in the C-norm from the mappings of ℘ by a positive constant. On the other hand, if G is totally disconnected (i.e., G is nowhere dense) then the class ℘ of convex mappings turns out to be stable in the sense that each mapping f which is approximated locally in neighborhoods of points in the domain dom f with high accuracy by mappings in ℘ is close in the C-norm to some mapping in ℘ over the whole interval dom f.

We borrow the exact definition of stability (see § 1) from the ω-stability concept [1], a formal theory agreeing well with the above intuitive arguments. In this article we give an exhaustive classification for ω-stable classes of Lipschitz mappings on intervals of the real axis R with values in R^m, m ≥ 1. It turns out that each ω-stable class ℘ is generated by some compact set G ⊂ R^m and partial preorder π on G by the following rule: ℘ consists of all Lipschitz mappings g : Δ ⊂ R → R^m such that g′(x) ∈ G a.e. and the derivative g′ increases with respect to π (Theorem 1). Moreover, the preorder π satisfies the decomposition condition below (see Definition 1). This in turn imposes some constraint on the geometric properties of the compact set G (see Proposition 1). Observe that in the above example of a class of convex functions the role of π is played by the conventional order on R.

Using this complete description for the ω-stable classes of mappings of intervals in R to R^m, we prove that, for all such classes, stability estimates hold in the C-norm as well as in the W^1_∞-norm (Theorem 2). Obtaining these estimates represents an important problem in stability theory. Here we recall the classical results by Reshetnyak [2] on stability in the W^1_p-norm of the classes of conformal and isometric transformations.

Stability estimates in the W^1_p-norm have been earlier known in ω-stability theory only for some special mapping classes, for example, for the above-mentioned classes of isometric mappings, some other classes of affine mappings, and classes of solutions to an elliptic system with constant coefficients (see [1–4]).
The results of this article were partially announced in [5]. For other results of \( \omega \)-stability theory see, for example, [4, 6, 7].

The article is divided into four sections. In §1 we recall the basic definitions of the \( \omega \)-stability concept; in §2 we state the main results and discuss the conditions in them; in §3 we give some typical examples; and in §4 we prove the theorems of §2.

§ 1. Definitions and Notations

Let \( n \) and \( m \) be arbitrary natural numbers (in this article we are interested in the case of \( n = 1 \)). A class \( \mathfrak{G} = \{ g : \Delta \subset \mathbb{R}^n \to \mathbb{R}^m \} \) of mappings of domains (open connected sets) \( \Delta \) of \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is normal if it satisfies the following Kopylov’s normality conditions [1].

\( K_1^\ast \). The class \( \mathfrak{G} \) consists of locally \( C \)-Lipschitz mappings with a fixed Lipschitz constant \( C = C_\mathfrak{G} \geq 0 \).

\( K_2^\ast \). If a mapping \( g : \Delta \subset \mathbb{R}^n \to \mathbb{R}^m \) belongs to \( \mathfrak{G} \), \( \rho > 0, a \in \mathbb{R}^n \), and \( b \in \mathbb{R}^m \) then the mapping \( g_0 : \Delta_0 \subset \mathbb{R}^n \to \mathbb{R}^m \) defined by the formula

\[
\Delta_0 \ni x \mapsto g_0(x) = \rho^{-1}g(\rho x + a) + b,
\]

where \( \Delta_0 = \{ \rho^{-1}(y - a) : y \in \Delta \} \), belongs to \( \mathfrak{G} \) too.

\( K_3^\ast \). The class \( \mathfrak{G} \) is closed with respect to locally uniform convergence.

\( K_4^\ast \). If \( g : \Delta \to \mathbb{R}^m \in \mathfrak{G} \) and \( \Delta_1 \subset \Delta \) is a subdomain of \( \Delta \) then \( g|_{\Delta_1} \in \mathfrak{G} \), where \( g|_{\Delta_1} \) is the restriction of \( g \) to \( \Delta_1 \).

\( K_5^\ast \). If a mapping \( g : \Delta \to \mathbb{R}^m \) is such that, for every \( x \in \Delta \), there is a neighborhood \( U(x) \subset \Delta \) for which \( g|_{U(x)} \in \mathfrak{G} \) then \( g \in \mathfrak{G} \).

The last two conditions mean that \( \mathfrak{G} \) is generated by some sheaf of mappings on \( \mathbb{R}^n \) with values in \( \mathbb{R}^m \).

A normal class \( \mathfrak{G} \) is called \( \omega \)-stable [1] (see also [8]) if there is a function \( \sigma : [0, +\infty) \to [0, +\infty) \) such that

1. \( \sigma(\varepsilon) \to \sigma(0) = 0 \) as \( \varepsilon \to 0 \);
2. the inequality \( \omega(f, \mathfrak{G}) \leq \sigma(\Omega(f, \mathfrak{G})) \) is valid for every mapping \( f : \Delta \to \mathbb{R}^m \) of the domain \( \Delta \subset \mathbb{R}^n \) with \( \Omega(f, \mathfrak{G}) < +\infty \).

Here \( \omega(f, \mathfrak{G}) = \sup_{B \subset \Delta} \omega_B(f, \mathfrak{G}) \) and \( \Omega(f, \mathfrak{G}) = \sup_{x \in \Delta} \Omega(x, f, \mathfrak{G}) \), where \( B = B(x, r) = \{ y \in \mathbb{R}^n : |y - x| < r \} \) is an \( n \)-dimensional ball in \( \Delta \), and

\[ \omega_B(f, \mathfrak{G}) = \inf_{g : B \to \mathbb{R}^m, g \in \mathfrak{G}} \{ r^{-1} \sup_{y \in B} |f(y) - g(y)| \}, \quad \Omega(x, f, \mathfrak{G}) = \lim_{r \to 0} \omega_B(x, r)(f, \mathfrak{G}). \]

These \( \omega(\cdot, \mathfrak{G}) \) and \( \Omega(\cdot, \mathfrak{G}) \) are the functionals of global and local proximity to \( \mathfrak{G} \). The global proximity functional \( \omega(\cdot, \mathfrak{G}) \) measures the proximity of a mapping \( f \) to the mappings of \( \mathfrak{G} \) over all balls in \( \Delta \), and the local proximity functional \( \Omega(\cdot, \mathfrak{G}) \) measures it only in infinitely small balls. Observe that \( \omega(f, \mathfrak{G}) = 0 \) amounts to \( f \in \mathfrak{G} \) (see [1]).

The notion of \( \omega \)-stability means essentially that a mapping \( f : \Delta \to \mathbb{R}^m \) can be approximated with high accuracy by mappings in \( \mathfrak{G} \) over every ball in \( \Delta \) if \( f \) is approximated with sufficient accuracy by mappings in \( \mathfrak{G} \) over infinitely small balls in \( \Delta \).

Given a set \( U \), henceforth \( \text{cl} U \) denotes the closure of \( U \) and \( \text{co} U \) is the convex hull of \( U \). We identify the space \( L(\mathbb{R}^n, \mathbb{R}^m) \) of linear mappings from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) with \( \mathbb{R}^{n \times m} \).

§ 2. Stability in the \( C \)-Norm and the \( W^{1, \infty} \)-Norm

The following notion plays an important part in the statements of the main theorems (see Theorems 1 and 2 below):

**Definition 1.** Suppose that \( G \subset \mathbb{R}^m \) is a nonempty compact set all whose connected components are convex. A partial preorder \( \pi \) on \( G \) is decomposable if the following hold for every pair \( a, b \in G \): If \( a \)