The Linnik conjecture is proved in the mean-square version with respect to the main parameter, namely, the length of the Linnik sum. Bibliography: 9 titles.

INTRODUCTION

In 1962, at the International Mathematical Congress in Stockholm Yu. V. Linnik (see [1]) advanced the conjecture on the double square reduction for the Kloosterman sums \( S(m, n; c) \) [2] in summing them over moduli \( c \geq 1 \):

\[
l_{m,n}(x) = \sum_{1 \leq c \leq x} \frac{S(m, n; c)}{c} \ll x^\varepsilon,
\]

(0.1)

where \( x \geq 1 \) is an increasing parameter; \( m \geq 1 \) and \( n \geq 1 \) are integer fixed parameters common for all Kloosterman sums; \( \varepsilon > 0 \) is an arbitrarily small but fixed number. The constant in estimate (0.1) depends on \( m \) and \( n \).

With the help of spectral techniques, in [5, 7] estimate (0.1) is obtained for \( \varepsilon = \frac{1}{6} \), which confirms the interference of the Kloosterman sums over the moduli, because the following estimate is trivial:

\[
l_{m,n}(x) \ll x^{1/2 + \varepsilon},
\]

which immediately follows from A. Weil’s estimate [3] for Kloosterman sums:

\[
|S(m, n; c)| \leq \sqrt{c} \cdot \min \left[ \sqrt{\frac{c}{(m, c)}}, \sqrt{\frac{c}{(n, c)}} \right].
\]

(0.2)

If we assume that the parameters \( m \) and \( n \) are fixed and \( c \) is increasing, then (0.2) can be written in a simpler form:

\[
S(m, n; c) \ll c^{1/2 + \varepsilon}.
\]

From this estimate it is seen that the Linnik conjecture suggests the second square reduction for the sum \( l_{n,n}(x) \) of Kloosterman sums over moduli \( c \leq x \).

In fact, estimate (0.1) carries two estimates: an analytic one with respect to the length \( x \) of the Linnik sum and an arithmetic one with respect to the parameters \( m \) and \( n \). In practice, these parameters are most often not fixed and increase with the length \( x \) of the interval. In order to clarify the form of estimate (0.1) with respect to three parameters \( (m, n, x) \), we use the following identity for Kloosterman sums (see [5]):

\[
S(m, n; c) = \sum_{\Delta \mid (m, n, c)} \Delta S(1, \frac{m \cdot n}{\Delta^2}, \frac{c}{\Delta}).
\]

(0.3)

Substituting (0.3) into the definition of the Linnik sum \( l_{m,n}(x) \) in (0.1), we obtain the identity

\[
l_{m,n}(x) = \sum_{\Delta \mid (m, n)} l_{1, \frac{m \cdot n}{\Delta^2}} \left( \frac{x}{\Delta} \right).
\]

(0.4)

From this identity, it follows that, without loss of generality of estimate (0.1), we may restrict ourselves to the Linnik sum with only one parameter:

\[
l_{1,m}(x) = \sum_{1 \leq c \leq x} \frac{S(1, m; c)}{c}.
\]

(0.5)

For this sum, the Linnik conjecture (0.1) can be written in the absolute form
\[ l_{1,m}(x) \ll (m \cdot x)^{\varepsilon}, \quad (0.6) \]
in the sense that the constant in (0.6) depends only on the parameter \( \varepsilon > 0 \). In this case, the Weil estimate (0.2) is simplified:
\[ |S(1, m; c)| \leq \sqrt{c} \cdot \tau(c). \quad (0.7) \]
If we substitute estimate (0.7) into (0.3), then the general Weil estimate (0.2) is also simplified:
\[ |S(m, n; c)| \leq \sqrt{c} \sum_{\Delta | (m, n') \neq e} \sqrt{\Delta} \cdot \tau \left( \frac{c}{\Delta} \right). \quad (0.2^a) \]
In the present paper, we prove the mean-square Linnik conjecture for sum (0.5):
\[ \int_{\mathbf{Z}}^{N} |l_{1,m}(x)|^2 dx \ll N^{1+\varepsilon}. \quad (0.8) \]
Moreover, the constant in estimate (0.8) does not depend on \( m \), provided that
\[ m \ll N^2. \quad (0.9) \]
We also obtain an arithmetic version of estimate (0.8) with respect to the parameter \( m \):
\[ \sum_{m \leq M} |l_{1,m}(x)|^2 \ll M^{1+\varepsilon}. \quad (0.10) \]
The constant in estimate (0.10) does not depend on \( x \) if the following condition holds:
\[ x \ll M \ll x^2. \quad (0.11) \]
One may combine estimates (0.8) and (0.10) if one considers the normed Linnik sum of the form
\[ l_{1,m}^*(x) = \sum_{c \leq 2 \sqrt{m \cdot x}} \frac{S(1, m; c)}{c}. \quad (0.12) \]
For the combined mean of this sum, we have
\[ \sum_{m \leq M} \int_{\mathbf{Z}}^{N} |l_{1,m}^*(x)|^2 dx \ll (M \cdot N)^{1+\varepsilon}. \quad (0.13) \]
The constant in estimate (0.13) depends only on \( \varepsilon > 0 \) if the condition
\[ M \ll N^2 \quad (0.14) \]
holds.

1. Preliminary information

In relation to the Linnik conjecture [1], in [4] A. Selberg introduced the \( \mathcal{Z} \)-function
\[ Z_{m,n}(S) = \sum_{c=1}^{\infty} \frac{S(m, n; c)}{c^{2s}}, \quad \text{Re } s > \frac{3}{4}, \quad (1.1) \]
and obtained the following spectral expansion for this function:
\[ Z_{m,n}(s) = (Z_{m,n}^{\text{con}} + Z_{m,n}^{\text{cusp}} + Z_{m,n}^{\text{disc}})(s). \quad (1.2) \]