New multivalue methods for
differential algebraic equations

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Multivalue methods are slightly different from the general linear methods John Butcher proposed over 30 years ago. Multivalue methods capable of solving differential algebraic equations have not been developed. In this paper, we have constructed three new multivalue methods for solving DAEs of index 1, 2 or 3, which include multistep methods and multistage methods as special cases. The concept of stiff accuracy will be introduced and convergence results will be given based on the stage order of the methods. These new methods have the diagonal implicit property and thus are cheap to implement and will have order 2 or more for both the differential and algebraic components. We have implemented these methods with fixed step size and they are shown to be very successful on a variety of problems. Some numerical experiments with these methods are presented.

**Keywords:** multivalue methods, differential algebraic systems of index 1, 2 or 3, A-stable, stiff accuracy, diagonal implicitness

1. Introduction

The purpose of this paper is to introduce a class of newly constructed multivalue methods for solving differential algebraic equations (DAEs) of Hessenberg like structure. Multivalue methods were introduced as a unifying tool for theoretical studies of many classes of numerical methods that have been studied independently of one another. Our study leads to new classes of methods, some of which appear to be very promising as solvers of DAE problems. The methods described in this paper are multistep Runge–Kutta methods (MRKs) which are concerned with solving index 1 or index 2 DAEs in the following forms.

Consider now the DAE of the form

\begin{align*}
y' & = f(y, z), \quad y(x_0) = y_0, \quad (1a) \\
0 & = g(y, z), \quad z(x_0) = z_0, \quad (1b)
\end{align*}
for \( x \in [x_0, x_f] \) with consistent initial values \( g(y_0, z_0) = 0 \). Equation (1) is called a differential algebraic equation or DAE, since it combines a differential equation (1a) with an algebraic equation (1b).

Let us define \( g_z \) as the matrix of partial derivatives \( \partial g / \partial z \). If \( g_z \) is nonsingular, it has bounded inverse in the neighbourhood of the solution of (1). Then (1) is said to be of index 1.

Now consider a problem of the form

\[
\begin{align*}
y' &= f(y, z), \\
y(x_0) &= y_0, \\
0 &= g(y), \\
z(x_0) &= z_0,
\end{align*}
\]

with consistent initial values \( g(y_0) = 0, \ g_z(y_0)f(y_0, z_0) = 0 \) and \( g_z(y)f_z(y, z) \) nonsingular in a neighbourhood of the solution of (2). Then (2) is said to be of index 2.

Finally, consider a DAE of the form

\[
\begin{align*}
y' &= f(y, z), \\
z' &= k(y, z, u), \\
0 &= g(y).
\end{align*}
\]

Differentiating (3c) with respect to time three times leads to

\[
0 = g_{yy}f_y f_y f_y + 3g_{yy}f_{yy}f_y + g_z f_{yy} + 2g_z f_{yz}f' + g_y f_{yy} + g_y f_{yz} + g_z f_{yz} + g_z f_{z} \frac{du}{dz}.
\]

If \( g_z f_{yz} k_u \) is nonsingular and \( y'' = f_y f_y + f_{yz} z' \) exists, the implicit ODE results, we say that the original problem has index 3.

### 1.1. Examples of DAEs

In this section, we demonstrate how we obtain the index 2 formulation of the discharge pressure control problem and how to prove this is a system of index 2. Before we move on, we first give the definition of the incidence matrix \( H \) of the algebraic system as

\[
H_{ij} = \begin{cases} 
1 & \text{if } z_j \text{ exists in } g_i, \\
0 & \text{otherwise}.
\end{cases}
\]

Here \( g_i \) and \( z_j \) represent the algebraic equations and algebraic variables respectively. The detailed analysis of incidence matrices can be found in [7]. To construct the incidence matrix, we need to identify which variables occur in each algebraic equation.

To study the index of the discharge pressure control problem [9], we introduce the auxiliary equation

\[
s = c + \frac{1}{15} p.
\]