Isometric Stochastic Flows on Spheres\textsuperscript{1}

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We consider isometric stochastic flows on the sphere $S^{n-1}$ with the same one point motion. In particular, we will show that when $n > 3$, the set of such flows with Brownian motion as one point motion can be represented by a cube in some Euclidean space.

KEY WORDS: Brownian motion; stochastic flows.

1. INTRODUCTION

Let $S^{n-1}$ be the $(n - 1)$-dimensional sphere considered as the unit sphere embedded in $R^n$. Suppose it is subject to random perturbation. The motion of the sphere can be described by a stochastic flow $g_t$ consisting of isometric transformations on $S^{n-1}$, hence will be called an isometric stochastic flow, whereas the motion of a fixed point on the sphere is a diffusion process $x_t$ on $S^{n-1}$. We are concerned with the following question: Does the motion of a point on the sphere determine the motion of the sphere? More precisely, can two different isometric stochastic flows on $S^{n-1}$ have the same one point motion? Here, we identify two stochastic processes when their distributions are the same.

All the isometric transformations on $S^{n-1}$ form the orthogonal group $O(n)$ whose identity component is $SO(n)$. The stochastic flow $g_t$ may be regarded as a right invariant diffusion process in $SO(n)$ starting at the identity element. Here, the right invariance of $g_t$ means $\forall g \in SO(n)$, the process $g_t g$ has the same distribution as $g_t$ starting at $g_0 g$. Recall the term diffusion process refers to a family of processes with arbitrary starting points.

When $n = 3$, we have proved in Liao\textsuperscript{4} that the one point motion uniquely determines the motion of the sphere. The solution is based on the

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fact that $SO(3)$ is a rank one group so that the adjoint action is transitive on the unit sphere in the Lie algebra of $SO(3)$. In the present paper, we will see that in general there are infinitely many isometric stochastic flows with the same one point motion on $S^{n-1}$ for $n > 3$, and in particular we will show that the set of isometric stochastic flows on $S^{n-1}$ with the Brownian motion as one point motion can be represented by a cube in some Euclidean space with its center representing the Brownian motion in $SO(n)$ and boundary points representing the degenerate flows.

We can state a more general version of this question. Let $M$ be a (smooth) manifold and let $G$ be a Lie transformation group on $M$ with identity element $e$. A stochastic flow on $M$ consisting of transformations in $G$ is a right invariant diffusion process $g_t$ in $G$ with $g_0 = e$, which will also be called a $G$-flow. We may ask whether two different $G$-flows can have the same one point motion.

Any right invariant differential operator on $G$ induces a differential operator on $M$. In particular, the generator of $g_t$ induces the generator of the one point motion of $g_t$. Since the generator determines the distribution of the process, our question can be rephrased as whether two different right invariant diffusion generators on $G$ can induce the same operator on $M$.

Some basic definitions will be given more precisely in the next section for general $M$ and $G$. Part of this section is taken from Liao.\(^\text{(4)}\) We will follow the ideas in Section II.2 of Helgason\(^\text{(1)}\) to introduce polynomial functions to represent right invariant differential operators on $G$. In Section 3, we will determine the space of quadratic polynomials which induce the zero operator on $S^{n-1}$ for $G = SO(n)$, and as a consequence, we will establish the previously mentioned representation of the set of $G$-flows of the Brownian motion in $S^{n-1}$ by a cube. In the last section, we will consider the situation on $S^3$ with $G = SO(4)$. In this case, the cube becomes the interval $[-1, 1]$ and the degenerate flow represented by 1 lies in a subgroup of $SO(4)$ which can be identified with $S^3$, in particular, the Brownian motion in $S^3$ becomes an isometric stochastic flow when $S^3$ is regarded as a subgroup of $SO(4)$.

2. SOME GENERAL DISCUSSION

Let $M$ be a (smooth) $d$-dimensional manifold and denote by $\mathcal{D}(M)$ the space of smooth functions on $M$. A differential operator $A$ on $M$ is the generator of a diffusion process $x_t$ on $M$ if it has the following expression under local coordinates $(x^1, x^2, \ldots, x^d)$.

$$A = \sum_{j,k=1}^{d} a_{jk}(x) \frac{\partial^2}{\partial x^j \partial x^k} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x^i}$$