On the Distribution of the Limit of Products of I.I.D. 2×2 Random Stochastic Matrices

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This article gives sufficient conditions for the limit distribution of products of i.i.d. 2×2 random stochastic matrices to be continuous singular, when the support of the distribution of the individual random matrices is finite.

KEY WORDS: Random matrices; limit distribution.

1. INTRODUCTION

In the 1960s and 1970s, Rosenblatt and others studied convergence of random walks taking values on compact semigroups [see Mukherjea and Tserpes⁶; Rosenblatt⁶], and properties of the limiting distributions in this abstract setting. However, very few concrete examples have been studied so far to illustrate these results. The set of 2×2 stochastic matrices, though an extremely simple (multiplicative) compact semigroup, is large enough to support some highly nontrivial cases that can shed light on a number of important questions in the above context. In this paper we study the question of continuous singularity of the limiting distribution, and then, as an application, consider the question whether the limiting distribution can arise from more than one random walk.

Let \( A_1, A_2, \ldots, A_n \) be 2×2 stochastic matrices such that the first column of \( A_i \) is \((x_i, y_i)\), where \(0 \leq x_i \leq 1, 0 \leq y_i \leq 1\). In this paper, the point \((x_i, y_i)\) on the plane will always represent the matrix \(A_i\). The second column of \(A_i\) is, of course, \((1-x_i, 1-y_i)\). Let \(\mu\) be a probability measure.

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such that the support of $\mu$ is given by $S(\mu) = \{A_1, A_2, \ldots, A_n\}$ and $\mu(A_i) = p_i$, $1 \leq i \leq n$. Let us assume that for each $i$, $1 \leq i \leq n$, $0 < x_i < 1$ and $0 < y_i < 1$. Then it is well-known [see Rosenblatt(6)] that the sequence $(\mu^n)$ of convolution powers of $\mu$ (where $\mu^n$ is the distribution of $Y_1 \cdot Y_2 \cdots Y_n$, $Y_i$'s being i.i.d. random matrices with distribution $\mu$) converges weakly to a probability measure $\lambda$ whose support consists of $2 \times 2$ stochastic matrices with identical rows, so that they are represented by points $(x, x)$, $0 \leq x \leq 1$.

Let us define the function $G(x)$ by

$$G(x) = \lambda\{ (y, y) : y \leq x \}$$

Then $G$ is the distribution function of $\lambda$, and since $\lambda * \mu = \lambda$, $G$ satisfies the functional equation:

$$G(x) = \sum_{i \in \mathcal{A}} p_i G\left(\frac{x - y_i}{x_i - y_i}\right) + \sum_{i \in \mathcal{B}} p_i \left[1 - G\left(\frac{x - y_i}{x_i - y_i}\right)\right] + \sum_{i \in \mathcal{C}\setminus x} p_i$$

(1.1)

where $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = \{0, 1, \ldots, n-1\}$, and the sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{C}[x]$ are given by

$$\mathcal{A} = \{i \mid x_i > y_i\}, \quad \mathcal{B} = \{i \mid x_i < y_i\}, \quad \mathcal{C} = \{i \mid x_i = y_i\}$$

and

$$\mathcal{C}[x] = \{i \mid x_i = y_i \leq x\}$$

For the purposes of this paper we will assume that $\mathcal{C}$ is empty. Indeed if $\mathcal{C} = \{1, 2, \ldots, n-1\}$, then $\lambda = \mu$. If, on the other hand, $\mathcal{C}$ is a nonempty proper subset of $\{0, \ldots, n-1\}$, the support of $\lambda$ is enumerable. Indeed, if

$$\mathcal{D}_0 = \{x \mid x = x_i \text{ for some } i \in \mathcal{C}\}, \quad \text{and}$$

$$\mathcal{D}_{m+1} = \mathcal{D}_m \cup \left\{x \mid x = x_i - y_i \text{ for some } i \in \mathcal{A} \cup \mathcal{B} \right\}, \quad m = 0, 1, 2, \ldots$$

(1.2)

and if

$$P = \sum_{i \in \mathcal{C}} p_i$$

(1.3)