On Positive Random Objects

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The idea of defining the expectation of a random variable as its integral with respect to a probability measure is extended to certain lattice-valued random objects and basic results of integration theory are generalized. Conditional expectation is defined and its properties are developed. Lattice valued martingales are also studied and convergence of sub- and supermartingales and the Optional Sampling Theorem are proved. A martingale proof of the Strong Law of Large Numbers is given. An extension of the lattice is also studied. Studies of some applications, such as on random compact convex sets in $\mathbb{R}^n$ and on random positive upper semicontinuous functions, are carried out, where the generalized integral is compared with the classical definition. The results are also extended to the case where the probability measure is replaced by a $\sigma$-finite measure.

KEY WORDS: Law of large numbers; optional sampling theorem; martingale convergence theorem.

1. INTRODUCTION AND PRELIMINARIES

The aim of this paper is to generalize the concept of the expectation of a positive random variable to other random objects. A few examples of such objects are

(a) $\mathbb{R}_+^n$-valued random vectors,
(b) Random positive upper semicontinuous functions on a compact topological Hausdorff space,
(c) Compact and convex random sets in $\mathbb{R}^n$, which contain the origin,

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where $R$ is the set of real numbers and $R_+$ is the set of nonnegative real numbers.

In (a), the expectation $E(X_1, \ldots, X_n)$ is normally defined as $(EX_1, \ldots, EX_n)$, where the $EX_i$'s are defined as the Lebesgue integrals, $\int_{\Omega} X_i \, dP$, where $\Omega$ is the sample space and $P$ the probability measure.

In (b), a natural way to define the expectation, $E_\xi$ of the random u.s.c. function $\xi$, would be to let $(E_\xi(\cdot)) = E[\xi(\cdot)]$ under the condition that this function is also u.s.c. This is for example known to be the case when $\xi$ is bounded by an integrable random variable, a fact which follows easily from Fatou's Lemma.

In the example (c), let $K$ be a random compact convex set containing the origin. One defines $EK$ as $\int_{\Omega} K \, dP$, where this integral is the integral for set-valued functions defined by Aumann.

Since we intend to try to define the expectation of a random object taking on values in some space of which these motivating examples are special cases, a natural start is to take a look at what these have in common. First of all, they take on values in a lattice, i.e., a space which, in a natural way, is partially ordered, and where every pair, $(x, y)$, of elements has a least upper bound, $\text{sup}(x, y)$, and a greatest lower bound, $\text{inf}(x, y)$. In the examples (a) and (b) before we use pointwise order and pointwise minima and maxima. In (c) we say that $K_1 \leq K_2$ if $K_1 \subseteq K_2$. Then $\text{inf}(K_1, K_2) = K_1 \cap K_2$ and $\text{sup}(K_1, K_2) = \bigcap\{K \in C(K_0(R^n)) : K \supseteq K_1 \cup K_2\}$. These examples give good reasons to begin the investigations with a brief summary of some general theory of partially ordered sets. This is done in Section 1.2 where we also introduce the Lawson topology and a few facts about it.

In Section 1.3, a general model for the space, $L$, of random objects is introduced and a few basic facts about the model are observed.

In Section 2.1, an $L$-valued random object is defined as an $L$-valued function and in Section 2.2, we define integrals of such functions. The Decreasing Monotone Convergence Theorem is proved along with the additivity of the integral. An analogue to Fubini's Theorem is proved for bounded functions. We also discuss why results such as the ordinary Increasing Monotone Convergence Theorem and the general Fubini's Theorem cannot be proved without further knowledge of $L$. We conclude the chapter with the definition of the expectation of an $L$-valued random object.

In Section 2.3, the definition of the integral is extended to $\sigma$-finite measure spaces. A closer study of the integral applied to some special cases is carried out in Section 2.4.

In Section 2.5 we metricize $L$ and make a few natural assumptions about $(L, d)$. Under these assumptions, we show that the Increasing