ON UNIFORMLY GÅTEAUX SMOOTH $C^{(n)}$-SMOOTH NORMS ON SEPARABLE BANACH SPACES

MARIÁN FABIÁN* and VÁCLAV ZIZLER,† Praha

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Abstract. Every separable Banach space with $C^{(n)}$-smooth norm (Lipschitz bump function) admits an equivalent norm (a Lipschitz bump function) which is both uniformly Gâteaux smooth and $C^{(n)}$-smooth. If a Banach space admits a uniformly Gâteaux smooth bump function, then it admits an equivalent uniformly Gâteaux smooth norm.

Let $(X, \| \cdot \|)$ be a separable Banach space. Then it is easy to construct an equivalent uniformly Gâteaux smooth norm on it. Indeed, let $\{x_j: j \in \mathbb{N}\}$ be a countable set contained and dense in the unit ball of $X$. Then

$$\|x^*\|^2 = \|x^*\|^2 + \sum_{j=1}^{\infty} x^*(x_j)^2/2^j,$$

is easily seen to be an equivalent, dual, and weak* uniformly rotund norm on $X^*$. Hence the corresponding norm $\| \cdot \|$ on $X$ is uniformly Gâteaux smooth. For more details see [DGZ, Section II.6]. Now assume that $X$ admits an equivalent $C^{(n)}$-smooth norm. A natural question then is whether $X$ admits an equivalent norm such that this norm would be both uniformly Gâteaux smooth and $C^{(n)}$-smooth. If $n = 1$, then $X^*$ is separable [Ph, Corollary 4.15, Theorem 2.19] and we can assume that the dual norm $\| \cdot \|$ on $X^*$ is locally uniformly rotund. Then the norm $\| \cdot \|$ on $X$ constructed above is both uniformly Gâteaux smooth and $C^{(1)}$-smooth. However, if $n > 1$, we seriously doubt that $\| \cdot \|$ would be $C^{(n)}$-smooth provided that $\| \cdot \|$ is.

The aim of this note is to construct such a norm:

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Theorem 1. Let \((X, \| \cdot \|)\) be a separable Banach space admitting an equivalent \(C^n\)-smooth norm, where \(n \in \{1, 2, \ldots\} \cup \{\infty\}\). Then \(X\) admits an equivalent norm which is both uniformly Gâteaux smooth and \(C^n\)-smooth.

We start with some preliminaries. The sets of positive integers, and real numbers are denoted by \(\mathbb{N}\), and \(\mathbb{R}\), respectively. Let \((X, \| \cdot \|)\) and \((Y, \| \cdot \|)\) be Banach spaces and \(n \in \mathbb{N}\). The symbol \(\mathcal{L}^n(X, Y)\) denotes the (Banach) space of \(n\)-linear bounded mappings from \(X\) to \(Y\) endowed with the norm

\[
\|L\| = \sup\{\|L(h_1, \ldots, h_n)\| : h_1, \ldots, h_n \in B_X\}, \quad L \in \mathcal{L}^n(X, Y).
\]

If \(n = 1\) we write \(\mathcal{L}(X, Y)\). If \(Y = \mathbb{R}\) we simply write \(\mathcal{L}^n(X) = \mathcal{L}(X, \mathbb{R})\). We use the symbol \(X^*\) instead of \(\mathcal{L}(X)\). The closed and open unit balls in \(X\) are denoted by \(B_X\) and \(B_X^*\) respectively.

Let \(f\) be a mapping from \(X\) to \(Y\) and \(x \in X\). We say that \(f\) is Gâteaux differentiable at \(x\) if there exists \(L \in \mathcal{L}(X, Y)\) such that

\[
\left\| \frac{1}{\tau} [f(x + \tau h) - f(x)] - L(h) \right\| \to 0 \quad \text{as} \quad \tau \to 0
\]

for every \(h \in X\). Then we denote \(f'(x) = L\). Let \(\Omega\) be an open subset in \(X\). We say that \(f\) is uniformly Gâteaux smooth on \(\Omega\) if \(f\) is Gâteaux differentiable at every point in \(\Omega\) and for every \(h \in X\)

\[
\left\| \frac{1}{\tau} [f(x + \tau h) - f(x)] - f'(x)(h) \right\| \to 0 \quad \text{as} \quad \tau \to 0
\]

uniformly for \(x \in \Omega\). It is easy to check that if \(f\) is Lipschitz on \(\Omega\), then \(f\) is uniformly Gâteaux smooth on \(\Omega\) if and only if \(f\) is Gâteaux differentiable at every point of \(\Omega\) and for every \(\varepsilon > 0\) and every \(h \in X\) there exists \(\delta > 0\) such that

\[
\|f'(x)(h) - f'(z)(h)\| < \varepsilon
\]

whenever \(x, z \in \Omega\) and \(\|x - z\| < \delta\). We say that the norm \(\| \cdot \|\) is uniformly Gâteaux smooth if it is uniformly Gâteaux smooth on the set \(\{x \in X : \|x\| > r\}\) where \(r\) is some (actually any) positive number.

We say that \(f\) is 1-times Fréchet differentiable at \(x\) if it is Gâteaux differentiable at \(x\) and

\[
\left\| \frac{1}{\tau} [f(x + \tau h) - f(x)] - f'(x)(h) \right\| \to 0 \quad \text{as} \quad \tau \to 0
\]

uniformly for \(h \in B_X\). Now let \(n \in \{2, 3, \ldots\}\) and assume that we have already defined the \((n - 1)\)-times Fréchet differentiability and the symbol \(f^{(n-1)}(x) \in \mathcal{L}^{n-1}(X, Y)\). We use the symbol \(X^*\) instead of \(\mathcal{L}(X)\). The closed and open unit balls in \(X\) are denoted by \(B_X\) and \(B_X^*\) respectively.