Surrogate Gradient Algorithm for Lagrangian Relaxation\textsuperscript{1,2}

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Abstract. The subgradient method is used frequently to optimize dual functions in Lagrangian relaxation for separable integer programming problems. In the method, all subproblems must be solved optimally to obtain a subgradient direction. In this paper, the surrogate subgradient method is developed, where a proper direction can be obtained without solving optimally all the subproblems. In fact, only an approximate optimization of one subproblem is needed to get a proper surrogate subgradient direction, and the directions are smooth for problems of large size. The convergence of the algorithm is proved. Compared with methods that take effort to find better directions, this method can obtain good directions with much less effort and provides a new approach that is especially powerful for problems of very large size.

Key Words. Lagrangian relaxation, integer programming, surrogate subgradient, job shop scheduling.

1. Introduction

Playing a fundamental role in constrained optimization in economics and mathematics over the decades, Lagrangian relaxation is particularly powerful for the optimization of separable nonlinear programming problems or integer programming problems. In a nutshell, the key idea of the approach

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is decomposition and coordination, where decomposition is based on the separability of models, and coordination based on the pricing concept of a market economy. In this method, coupling constraints are first relaxed through the introduction of Lagrange multipliers, and the relaxed problem can be decomposed into many smaller subproblems. Given the set of multipliers, these subproblems are easier than the original problem, and can be solved efficiently. Multipliers are then adjusted iteratively based on the levels of constraint violation. The resolution of the original problem is thus done through a two-level iterative approach, where the low level consists of solving individual subproblems. Coordination of the subproblem solutions is performed through the updating of the Lagrange multipliers at the high level.

In the optimization terminology, the concave dual function is maximized iteratively. In this process, the subproblem solutions will tend to an optimal feasible solution, while the dual function itself provides a lower bound to the optimal primal cost (Ref. 1).

When applying Lagrangian relaxation to integer programming, techniques such as the subgradient, bundle, and cutting plane methods are used often to maximize the dual function, since the dual function is polyhedral concave and nondifferentiable. The subgradient method is the most widely used method, where the subgradient direction is obtained after all the subproblems are solved and the multipliers are updated along this subgradient direction. The bundle method can provide better directions than the subgradient method. However, to obtain each direction, it may require solving all the subproblems many times (Ref. 2). For problems of large size, the solution of the subproblems can be complicated and takes the majority of the computation time. For example, it has been reported that more than 70% of the total CPU time is spent on solving subproblems for job shop scheduling problems (Ref. 3). Therefore, it is desirable to obtain a good direction with less effort than solving all the subproblems many times to obtain an optimized direction.

Based on this idea, the interleaved subgradient method solves only one subproblem per iteration to obtain a direction and then updates the multipliers (Ref. 4). Numerical results show that the method converges faster than the subgradient method, although the algorithm convergence was not established.

In this paper, the surrogate subgradient method is developed, and a proper surrogate subgradient direction to update the multipliers can be obtained without solving all the subproblems. In fact, only an approximate solution of one subproblem is needed to obtain a surrogate subgradient direction. The convergence of the algorithm is proved in Section 4 for separable nonlinear programming or integer programming problems. This method includes the interleaved subgradient method as a special case and provides