TECHNICAL NOTE

Vector Variational Inequality and Vector Pseudolinear Optimization

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Abstract. The study of a vector variational inequality has been advanced because it has many applications in vector optimization problems and vector equilibrium flows. In this paper, we discuss relations between a solution of a vector variational inequality and a Pareto solution or a properly efficient solution of a vector optimization problem. We show that a vector variational inequality is a necessary and sufficient optimality condition for an efficient solution of the vector pseudolinear optimization problem.

Key Words. Vector variational inequalities, vector optimization, optimality conditions.

1. Introduction

Let

\[ f(x) := (f_1(x), \ldots, f_p(x))^T; \]

let the functions \( f_i : \mathbb{R}^n \to \mathbb{R} \) be scalar-valued and differentiable, \( i = 1, \ldots, p \); and let \( K \) be a convex subset of \( \mathbb{R}^n \). Let \( C = \mathbb{R}_+^p \) be the positive orthant of \( \mathbb{R}^p \), and define the ordering \( \preceq (\leq) \) in \( \mathbb{R}^p \) as follows:

for \( x, y \in \mathbb{R}^p \), \( x \preceq y \) [\( x \leq y \)] if and only if \( y - x \not\in C \setminus \{0\} \) [\( y - x \in C \setminus \{0\} \)].

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Consider the vector optimization problem
\[
\min_c f(x), \quad \text{s.t. } x \in K, \tag{1}
\]
where \(\min_c\) means vector optimization with respect to the closed convex cone \(C\). A point \(x \in K\) is said to be a Pareto (efficient) solution of (1) if there exists no \(y\) such that \(f(y) \leq f(x)\). A point \(x \in K\) is said to be a properly efficient solution of (1) if there exists a scalar \(M > 0\) such that, for each \(i\),
\[
[f_i(x) - f_i(y)]/[f_j(y) - f_j(x)] \leq M,
\]
for some \(j\) such that \(f_j(y) > f_j(x)\) whenever \(y \in K\) and \(f_i(y) < f_i(x)\). Every properly efficient solution is a Pareto solution.

Let
\[
f'(x) := (f'_1(x), \ldots, f'_p(x))^T
\]
be the Jacobian matrix \((p \times n)\) of the vector-valued function \(f\) at \(x\). Consider the following vector variational inequality (Ref. 1): Find \(x \in K\) such that
\[
f'(x)(y - x) \leq 0, \quad \forall y \in K. \tag{2}
\]
It is not difficult to show that the following proposition holds (Refs. 2 and 3).

**Proposition 1.1.** If \(x\) is a solution of (2) and \(f\) is convex, then \(x\) is a Pareto solution of (1).

It is also shown in Refs. 2 and 3 that in general (2) is not a necessary optimality condition for a Pareto solution of (1) by the following example.

**Example 1.1.** Consider the problem
\[
\min_c (f_1(x), f_2(x))^T, \quad \text{s.t. } x \in [-1, 0],
\]
where
\[
f_1(x) = x, \quad f_2(x) = x^2.
\]
It is clear that every \(x \in [-1, 0]\) is a Pareto solution of (1). Let \(x = 0\). Then, for \(y = -1\),
\[
(f_1'(x)(y - x), f_2'(x)(y - x))^T = (-1, 0)^T \leq (0, 0)^T.
\]
Thus, \(x = 0\) is not a solution of (2).

It is shown in Ref. 1 that the following proposition holds.