su_q(2)-IN Variant Harmonic Oscillator

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We propose a q-deformation of the su(2)-invariant Schrödinger equation of a spinless particle in a central potential, which allows us not only to determine a deformed spectrum and the corresponding eigenstates, as in other approaches, but also to calculate the expectation values of some physically-relevant operators. Here we consider the case of the isotropic harmonic oscillator and of the quadrupole operator governing its interaction with an external field. We obtain the spectrum and wave functions both for $q \in \mathbb{R}^+$ and generic $q \in S^1$, and study the effects of the q-value range and of the arbitrariness in thesu_q(2) Casimir operator choice. We then show that the quadrupole operator in $l = 0$ states provides a good measure of the deformation influence on the wave functions and on the Hilbert space spanned by them.

1 Introduction

Since the advent of quantum groups and quantum algebras, there has been a lot of interest in deformations of the harmonic oscillator, since the latter plays a central role in the investigation of many physical systems. Various q-deformed versions of standard quantum mechanics in the Schrödinger representation were proposed for the oscillator by using either the ordinary differentiation operator (see e.g. [1]), or a q-differentiation one (see e.g. [2]). Some works also involve non-commutative objects (see e.g. [3]).

In previous papers [4, 5] and in this talk, we set up an su_q(2)-invariant Schrödinger equation within the framework of quantum mechanics, using a representation of the su_q(2) quantum algebra on the two-dimensional sphere [6, 7], which allows us not only to determine a deformed spectrum, as in other works, but also to calculate the expectation values of some physically-relevant operators. In our approach, only the angular sector is deformed. This gives rise to an appropriate change in the angular part of the scalar product [7], and to the substitution of the su_q(2) Casimir operator for the su(2) one in the kinetic part of the Hamiltonian while the potential part remains unchanged. The latter step may be performed in various ways since there is no unique rule for constructing the su_q(2) Casimir operator. Similarly, the
deforming parameter $q$ may be assumed either real and positive, or on the unit circle in the complex plane (but different from a root of unity), provided different scalar products are used [7]. We will study the effects of these two choices on the spectrum and on the value of the quadrupole moment.

In Section 2, the Schrödinger equation of the $\text{su}_q(2)$-invariant harmonic oscillator is introduced and solved. In Section 3, its spectrum is studied in detail for various choices of $\text{su}_q(2)$ Casimir operators and $q$ ranges. The effect of the deformation on the corresponding wave functions is determined in Section 4 by calculating the quadrupole moment in $l = 0$ states. Finally, Section 5 contains the conclusion.

2 $\text{su}_q(2)$-invariant Schrödinger equation

Let

$$H_q = -\frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{C_q}{r^2} \right) + \frac{1}{2} \mu \omega^2 r^2$$

(1)

be the Hamiltonian of a $q$-deformed three-dimensional harmonic oscillator in spherical coordinates $r, \theta, \phi$. Here $C_q$ is the $\text{su}_q(2)$ Casimir operator, which we may take as

$$C_q = J_+ J_- + [J_3 - \frac{1}{2}]_q^2 - \frac{1}{4},$$

(2)

where $[x]_q \equiv (q^x - q^{-x}) / (q - q^{-1})$, and $q = e^{i\omega} \in \mathbb{R}^+$ or $q = e^{i\omega} \in \mathbb{S}^1$ (but different from a root of unity). The operators $J_3, J_+, J_-$, satisfying the $\text{su}_q(2)$ commutation relations

$$[J_3, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = [2J_0]_q,$$

(3)

are defined in terms of the angular variables by

$$J_3 = -i\partial_\phi,$$
$$J_+ = -e^{i\phi} \left( \tan(\frac{1}{2}\theta) [T_1]_q + \cot(\frac{1}{2}\theta) [T_2]_q \right),$$
$$J_- = e^{-i\phi} \left( \cot(\frac{1}{2}\theta) [T_1]_q + \tan(\frac{1}{2}\theta) [T_2]_q \right),$$

(4)

with $T_1 = -\frac{1}{2} (\sin \theta \partial_\theta - i \partial_\phi)$, $T_2 = -\frac{1}{2} (\sin \theta \partial_\theta + i \partial_\phi)$ [6, 4].

Instead of Eq. (2), we may alternatively use the operator

$$C'_q = J_+ J_- + [J_3]_q [J_3 - 1]_q$$

(5)

in Eq. (1), in which case the corresponding Hamiltonian will be denoted by $H'$. The Hamiltonians $H_q$ and $H'_q$ remain invariant under $\text{su}_q(2)$ since they commute with $J_3, J_+, J_-$, and they coincide with the Hamiltonian of the standard three-dimensional isotropic oscillator when $q = 1$. For simplicity's sake, we shall henceforth adopt units wherein $\hbar = \mu = \omega = 1$.

In this representation, the stationary wave functions $|nlm\rangle_q$ can be written as $R_{n\mu}(r) Y_{l\mu q}(\theta, \phi)$, where $Y_{l\mu q}(\theta, \phi)$ are the $q$-spherical harmonics introduced in [6]. We proved that they form an orthonormal set with respect to the scalar product [7].