NEW APPROACH IN THEORY OF CLEBSCH-GORDAN COEFFICIENTS FOR $u(n)$ AND $U_q(u(n))$\textsuperscript{*})

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A new method for calculation of Clebsch-Gordan coefficients (CGCs) of the Lie algebra $u(n)$ and its quantum analog $U_q(u(n))$ is developed. The method is based on the projection operator method in combination with the Wigner-Racah calculus for the subalgebra $u(n-1)$ ($U_q(u(n-1))$). The key formulas of the method are couplings of the tensor and projection operators and also a tensor form of the projection operator of $u(n)$ and $U_q(u(n))$. It is shown that the $U_q(u(n))$ CGCs can be presented in terms of the $U_q(u(n-1))$ $q$-9j-symbols.

1 Introduction

It is well known that the Clebsch-Gordan coefficients (CGCs) of the unitary Lie algebra $u(n)$ ($su(n)$) have numerous applications in different fields of the theoretical and mathematical physics. For example, many algebraic models of the nuclear theory such that: the interaction boson model (IBM), the Elliott $su(3)$-model, the $su(4)$-supermultiplet scheme of Wigner, the shell model and so on, demand explicit expressions of the CGCs for $su(6)$, $su(5)$, $su(3)$, $su(4)$ and $su(n)$. Analogously, in different quark models of the hadrons we need the CGCs of $su(3)$, $su(4)$ etc. The theory of the $su(n)$-CGCs is connected with the theory of special functions, the combinatorial analysis, topology, etc.

There are several methods for calculation of CGCs of $su(n)$ ($u(n)$) and other Lie algebras: recurrent method, method of employment of explicit bases of irreducible representations, method of generating invariants, method of tensor operators (where the Wigner-Eckart theorem is used), projection operator method, coherent state method, other combined methods.

We briefly illustrate structure of these methods for the case $su(2)$. Let $\{j_km_k\}$ be canonical bases of two irreducible $su(2)$ representations (IRs) $j_k$ ($k = 1, 2$). Then

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\[ \{j_{1m_1}j_{2m_2}\} \] is a canonical basis in the representation \( j_1 \otimes j_2 \) of \( su(2) \otimes su(2) \). In this representation there is another basis \( \{j_{1j_2}:j_{3m_3}\} \) which is called a reduced basis with respect to \( \Delta (su(2)) \). We can expand the reduced basis in terms of the tensor basis \( \{j_{1m_1}j_{2m_2}\} \):

\[
\{j_{1j_2}:j_{3m_3}\} = \sum_{m_1,m_2} \langle j_{1m_1}j_{2m_2}|j_{3m_3}\rangle \langle j_{1m_1}|j_{2m_2}\rangle, \tag{1}
\]

where the matrix element \( \langle j_{1m_1}j_{2m_2}|j_{3m_3}\rangle \) is the Clebsch-Gordan coefficient (CGC).

Let \( J_i^1 \) and \( J_i^2 \) be generators of the IRs \( j_1 \) and \( j_2 \), then \( J_i^{12} = J_i^1 + J_i^2 \) are generators of the resulting representation of \( \Delta (su(2)) \).

1. **The recurrent method.** Applying the operators \( J_i(12) \) to the relation (1) we obtain the following system of recurrent relations

\[
\langle j_{3m_3'}|J_i^{12}|j_{3m_3}\rangle (j_{1m_1}j_{2m_2}|j_{3m_3'}) = \\
\langle j_{1m_1}|J_i^1|j_{1m_1}'\rangle (j_{1m_1'}j_{2m_2}|j_{3m_3'}) + \langle j_{2m_2}|J_i^2|j_{2m_2}'\rangle (j_{1m_1}j_{2m_2'}|j_{3m_3'}). \tag{2}
\]

Solving this system we can obtain the explicit formula for the CGCs of \( su(2) \).

2. **The method of employment of explicit bases of irreducible representations.** If we know the explicit form of the basis vectors \( \{j_{1j_2}:j_{3m_3}\} \) then we can expand these vectors in the terms of the basis vectors \( \{j_{1m_1}j_{2m_2}\} \) and we obtain a explicit formula for the CGCs of \( su(2) \).

3. **In the method of generating invariants.** we construct the invariant of the type

\[
\begin{vmatrix}
  u_{11} & u_{12} & u_{13} \\
  u_{21} & u_{22} & u_{23} \\
  u_{31} & u_{32} & u_{33}
\end{vmatrix}
= \sum_{R_{ik}} R_{11} R_{12} R_{13} R_{21} R_{22} R_{23} R_{31} R_{32} R_{33} \prod_{k,l=1}^{3} \frac{v_{kl}^{R_{kl}}}{R_{kl}!}, \tag{3}
\]

where the R-symbol \( \|R_{ik}\| \) is connected with the CGC of \( su(2) \).

4. **The method of tensor operators.** If we know matrix elements of the irreducible tensor operators \( T_{ijm}^j \) then using the Wigner-Eckart theorem

\[
\langle j_{2m_2}|T_{ijm}^j|j_{1m_1}\rangle = \langle j_{1m_1}jm|j_{2m_2}\rangle \langle j_1\|T^j\|j_2 \rangle, \tag{4}
\]

we can obtain the CGCs.

5. **The projection operator method.** In this case CGCs are calculated by the formula

\[
\langle j_{1m_1}j_{2m_2}|j_{3m_3}\rangle = \frac{\langle j_{1m_1}|j_{2m_2}\Delta (P_{jj_3}^j)|j_{1j_1}\rangle |j_2j_3 - j_1 \rangle}{\langle j_{1j_1}|j_{2j_3} - j_1 \Delta (P_{jj_3}^j)|j_{1j_1}\rangle |j_2j_3 - j_1 \rangle^{1/2}}, \tag{5}
\]

where \( P_{jj_3}^j \) is the projection operator of \( su(2) \).