ON A SUBVARIETY OF SEMI-DE MORGAN ALGEBRAS

C. PALMA and R. SANTOS (Lisbon)

Abstract. A characterization of principal congruences of the subvariety $C$ of semi-De Morgan algebras is given. This characterization is applied to determine the subdirectly irreducible algebras of the variety $C$ and to describe a poset such that the lattice of its order ideals is isomorphic to the lattice of subvarieties of $C$.

1. Introduction

The equational class of semi-De Morgan algebras was introduced by Sankappanavar in [15]. It consists of bounded distributive lattices with an additional unary operation and it contains the variety of pseudo-complemented distributive lattices (also called p-lattices) and $K_{1,1}$, one of the subvarieties of Ockham algebras which includes De Morgan algebras.

Sankappanavar has continued the investigation of semi-De Morgan algebras in [16] and [17], where he concentrated on the class of demi-pseudo-complemented lattices, also called demi-p-lattices, an important subvariety of semi-De Morgan algebras which generalizes p-lattices.

In [9] Hobby developed a duality for semi-De Morgan algebras which is a Priestley-type duality for distributive lattices endowed with an additional binary relation. He used this duality to find the largest subvariety of semi-De Morgan algebras with the congruence extension property. This variety which Hobby denoted by $C$ contains both $K_{1,1}$ and the equational class of demi-p-lattices.

In this paper we study some properties of the variety $C$. In Section 3 we characterize the principal congruences on $C$, extending the corresponding characterization for demi-p-lattices due to Sankappanavar [16], and for the variety $K_{1,1}$ due to J. Berman [3] and to M. Ramalho and M. Sequeira

1Work done within the activities of CAUL and partly supported by FCT and "Programa Ciência, Tecnologia e Inovação do Quadro Comunitário de Apoio".

Key words and phrases: semi-De Morgan algebra, principal congruence, subdirectly irreducible algebra.

1991 Mathematics Subject Classification: primary: 06D15, 06D90, 06D99, secondary: 08A30, 08B26.

0236-5294/3/$20.00$ © 2003 Akadémiai Kiadó, Budapest
[12]. As an application, it is shown that $C$ has equationally definable principal congruences, a result which strengthens Hobby’s result that this variety has the congruence extension property. In Section 4 we characterize the subdirectly irreducible algebras of the variety $C$. The subdirectly irreducible demi-$p$-lattices were characterized by Sankappanavar in [16] and the subdirectly irreducible algebras of the variety $K_{1,1}$ were identified in [14] and also in [2]. We use these results and the characterization of principal congruences to prove that apart from the subdirectly irreducible algebras of the varieties of demi-$p$-lattices and $K_{1,1}$ there are, up to isomorphism, three more subdirectly irreducible algebras in $C$. We consider the set of isomorphism classes of finite subdirectly irreducible algebras of the variety $C$ under the ordering of homomorphic image of a subalgebra and we present the Hasse diagram of this poset. Using a theorem of B. Davey [6], we prove that the lattice of subvarieties of $C$ is isomorphic to the lattice of order-ideals of this poset. In Section 5 we give defining identities for some subvarieties of $C$.

2. Preliminaries

We start by recalling some definitions and essential results as well as some notation, previously adapted by other authors, which will be useful in the later sections.

**Definition 2.1.** An Ockham algebra is an algebra $(A, \lor, \land, ', 0, 1)$ for which $(A, \lor, \land, 0, 1)$ is a bounded distributive lattice satisfying the identities

$$(x \lor y)' \approx x' \land y', \quad (x \land y)' \approx x' \lor y', \quad 0' \approx 1 \quad \text{and} \quad 1' \approx 0.$$ 

The subvariety $K_{1,1}$ of the variety of Ockham algebras, first considered by J. Berman in [3], is the class of Ockham algebras which satisfy $x' \approx x'''$. An algebra of $K_{1,1}$ is a De Morgan algebra if and only if it satisfies $x' \approx x$. In what follows DUA will denote the equational class of De Morgan algebras.

The subdirectly irreducible algebras of the variety $K_{1,1}$ (sometimes also denoted by $P_{3,1}$) were obtained by Sankappanavar in [14] and independently by Beazer in [2]. Their diagrams are presented in [5] on pages 70 and 71 and the poset of these subdirectly irreducible algebras ordered according to a theorem of Davey [6] is presented in [5] on page 91.

For an easier understanding of this paper we recall that the subdirectly irreducible algebras of $K_{1,1}$ were denoted in [5] by: $B$, $K$, $M$, $S$, $\overline{S}$, $S_1$, $K_1$, $K_2$, $K_3$, $K_4$, $M_1$, $M_2$, $L$, $\overline{L}$, $N$, $\overline{N}$ and $B_1$.

It is well known that the subdirectly irreducible De Morgan algebras are $B$, $K$ and $M$ with diagrams as in Fig. 1.

**Definition 2.2.** An algebra $L = (L, \lor, \land, ', 0, 1)$ is a semi-De Morgan algebra if the following five conditions hold $(x, y \in L)$: