Polynomial Hurwitz Numbers and Intersections on $\overline{M}_{0,k}^*$

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Received April 12, 2002

Abstract. We express Hurwitz numbers of polynomials of arbitrary topological type in terms of intersection numbers on the moduli space of curves of genus zero with marked points.

Key words: Hurwitz numbers, polynomials, moduli space of curves, intersection numbers.

1. Consider a polynomial $P$ of degree $n$. Let $\Sigma = \{x_1, \ldots, x_m\}$ be a finite subset of $\mathbb{CP}^1$ containing all critical values of $P$ including infinity (let us put $x_1 = \infty$). The passport of a point $x_i \in \Sigma$ (with respect to the polynomial $P$) is the tuple of positive integers $A_i(P) = (a_i, \ldots, a_i^l)$ such that $P^{-1}(x_i) = a_i^1 y_1 + \cdots + a_i^l y_l$, where $y_1, \ldots, y_l$ are pairwise distinct points of the sphere $\mathbb{CP}^1$ and the sum is understood as a formal sum.

In particular, the passport $A_1(P)$ of the point $x_1 = \infty$ is equal to $(n)$. If $x_i$ is a regular value of $P$, then $A_i(P) = (1, \ldots, 1)$. If $x_i$ is a simple critical value of $P$, then $A_i(P) = (2, 1, \ldots, 1)$.

2. Let us assign a passport $A_i = (a_i^1, \ldots, a_i^l)$ to each point $x_i \in \Sigma$. Let $A_1 = (n)$ and suppose that $\sum_{j=1}^l a_i^j = n$ for any $i$ and $\sum_{i=1}^m (n - l_i) = 2n - 2$. These conditions are necessary for the existence of a polynomial $P$ such that all its critical values belong to $\Sigma$ and $A_i(P) = A_i$ for any $i = 1, \ldots, m$.

The Hurwitz number $h$ is the number of all such polynomials $P$. Here we consider polynomials up to the action of $PGL(2, \mathbb{C})$ on the source projective line. Each polynomial is counted with the weight $1/c$, where $c$ is the number of automorphisms of the polynomial.

There is a simple combinatorial formula for this number (see [5]). Our aim is to express this number via intersection numbers on $\overline{M}_{0,k}$. Our formula gives nothing for the problem of calculation of Hurwitz numbers, but such formulas seem to be interesting by themselves and sometimes they are very useful (see [1, 6]).

3. Let $\overline{M}_{0,k}$ denote the moduli space of stable curves of genus 0 with $k$ marked points. By $\psi(y)$ we denote the first Chern class of the line bundle over $\overline{M}_{0,k}$ whose fiber is the cotangent line at the marked point $y$.

We will consider intersections on the space $\overline{M}_{0,\sum_{i=1}^m l_i}$; the marked points are denoted by $y_j^i$, $i = 1, \ldots, m$, $j = 1, \ldots, l_i$.

By $\pi_{p,q,k}: \overline{M}_{0,\sum_{i=1}^m l_i} \rightarrow \overline{M}_{0,2+p}$ we denote the projection forgetting all marked points except for $y_1^1, y_1^p, \ldots, y_i^p$, and $y_k^q$. By $D_U$, $U \subset \{y_j^i\}$, we denote the cohomology class dual to the divisor whose generic point is represented by a two-component curve such that all elements of $U$ lie on one component and all elements of $\{y_j^i\} \setminus U$ lie on the other component.

Let us define classes $\psi_p(b_j^{i})_{j=1,\ldots,\sum_{i=1}^m l_i}$ by the following recursive relations. If $(b_j^{i}) = 0$, then $\psi_p(b_j^{i}) = 1$. Let the tuple of numbers $b_j^{i}$ be different from the tuple of numbers $b_j^{i}$ only in the term $i = q$, $j = k$, where $b_j^{i} = b_j^{i} + 1$. In this case,

$$\psi_p(b_j^{i}) = \psi_p^{*}(b_j^{i}) = \sum_{U} a_U \psi_p((b_j^{i})_{j=1,\ldots,\sum_{i=1}^m l_i}) D_{U \cup \{y_k^q\}} - \sum_{U} a_U \psi_p((b_j^{i})_{j=1,\ldots,\sum_{i=1}^m l_i}) D_{U \cup \{y_k^q\}},$$

where $\sum_{U}$ is understood as a formal sum.

*This work is partially supported by grants RFBR-01-01-00660 and INTAS-00-0250.
where the sum is taken over all \( U \subset \{ y_j^i \} \setminus \{ y_1^1, y_1^2, \ldots, y_m^p, y_2^0 \} \), \( a_U = \sum_{y_j^i \in U} b_j^i \), \((b_U)_j^i = b_j^i \) if \( y_j^i \notin U \cup \{ y_2^0 \} \), \((b_U)_j^i = 0 \) if \( y_j^i \in U \), and \((b_U)_j^i = a_U + b_j^i \). The correctness of this definition is a nontrivial corollary of Keel’s theorem, see [4]. We shall use only the classes \( \Psi_p = \Psi_p(a_j^i - 1) \), i.e., the classes defined by the tuples of numbers \((a_j^i - 1)_{j=1, \ldots, t_i} \).

4. Now we can state our theorem.

**Theorem 1.** For \( m \geq 3 \), the following formula is valid:

\[
h = \frac{n^{m-3}}{\prod_{i=1}^m \text{aut}(A_i)} \int_{\mathcal{M}_0 \Sigma_{i=1}^m \overline{t}_m} \psi(y_1^1)^{m-3} \prod_{p=2}^m \Psi_p.
\]

5. Let us show how one can verify this formula in the simplest cases.

**Case 1.** Let \( m = n \), \( A_1 = (n) \), and \( A_2 = \cdots = A_m = (2, 1, \ldots, 1) \). In other words, we consider the case, where all critical values are simple.

One can prove that, in this case, \( \Psi_p \) is dual to the subvariety whose generic point is represented by an \((n-1)\)-component curve. Moreover, \( y_1^1 \) and \( y_2^0 \) lie on the component that is intersected by all other \( n-2 \) components, and we have exactly one point \( y_1^1 \) and exactly one point \( y_2^0 \) on each of these \( n-2 \) components.

This subvariety has \( n-2 \) irreducible components. Therefore, the integral part of the formula is equal to \((n-2)! \)^{n-1} and \( h = n^{n-3} \).

**Case 2.** Let \( m = 3 \), \( A_1 = (n) \), \( A_2 = (a_2^2, a_2^3) \), and \( A_3 = (2, 1, \ldots, 1) \). Then we have

\[
\Psi_2 = \pi_{2,3,1}^* \psi(y_1^1),
\Psi_3 = (a_2^2 - 1)! (a_2^3 - 1)! \pi_{3,2,1}^* \psi(y_1^1)^{a_2^2 - 1} \pi_{3,2,2}^* \psi(y_2^0)^{a_2^3 - 1}
- ((n-2)! - (a_2^2 - 1)! (a_2^3 - 1)! \pi_{2,3,1}^* \psi(y_1^1)^{a_2^2 - 1}.
\]

Note that \( \Psi_2 \) is dual to the divisor whose generic point is represented by a two-component curve such that \( y_1^1 \) and \( y_2^0 \) lie on one component and \( y_2^2 \) and \( y_3^1 \) lie on the other component. The restriction of \( \Psi_3 \) to this divisor is equal to

\[
(a_2^2 - 1)! (a_2^3 - 1)! \psi(y_2^0)^{a_2^2 - 1} \psi(y_2^0)^{a_2^3 - 1}.
\]

Then, using standard calculations [4], we obtain that \( h = 1/\text{aut}(A_2) \).

6. Let us sketch the proof of the theorem.

Consider \( \Sigma \) as a point of \( \mathcal{M}_0 \). Instead of polynomials, we consider pairs \((P, \Sigma)\), where \( P \) is a polynomial, \( \Sigma \in \mathcal{M}_0 \), and all passports are fixed.

The mapping \( ll: (P, \Sigma) \mapsto \Sigma \) is a finite-sheeted covering over \( \mathcal{M}_0 \), and \( h \) is the degree of the covering.

The space of pairs \((P, \Sigma)\) admits the Harris–Mumford compactification \( \overline{H} \) (see [2]) such that the mapping \( ll \) extends to a ramified covering \( ll: \overline{H} \mapsto \mathcal{M}_0 \).

Since \( \int_{\mathcal{M}_0} \psi(x_1)^{m-3} = 1 \), we have \( h = \int_{\overline{H}} (ll^* \psi(x_1))^{m-3} \).

Let us mark all preimages of each point in \( \Sigma \). Consider the mapping \( st: \overline{H} \mapsto \mathcal{M}_0 \Sigma_{i=1}^m \overline{t}_i \) that takes a pair \((P, \Sigma)\) to the source curve of \( P \) on which all preimages of each point in \( \Sigma \) are marked.

By the lemma in [3], \( n \cdot st^* \psi(y_1^1) = ll^* \psi(x_1) \) (if \( y_1^1 \) is a preimage of \( x_1 \) of multiplicity \( n \)). Therefore, \( h = \int_{\overline{H}} (n \cdot st^* \psi(y_1^1))^{m-3} / (\prod_{i=1}^m \text{aut}(A_i)) \) (here we also divide by the number of possible markings of all preimages for all points in \( \Sigma \)).

In order to obtain formula (2), it remains to calculate the class of \( st_\* \overline{H} \). In order to do this, it is sufficient to express in terms of intersections the requirement that some marked points be critical points of a polynomial with prescribed ramification indices and the values of this polynomial at some marked points coincide.

7. The author is grateful to S. K. Lando, S. M. Natanzon, and M. Z. Shapiro for useful discussions. For the complete proof of the theorem, see [7].