Properties of Vortex Crystals in BCS Superconductors by means of the Expansion in the "Distance" from the $H_{c2}(T)$ Line

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We develop a description of the mixed state of type II superconductivity valid within a wide range of temperatures and external magnetic fields. It is based on the quasiclassical version of microscopic BCS theory and employs an expansion in the parameter that can be regarded as the "distance" from the $H_{c2}(T)$ line to a given point on the $T$ – $B$ plane.

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In this paper we present a rigorous perturbative method for description of the mixed state of BCS superconductors. To find the structure of a vortex solid at an arbitrary $(T, B)$ point directly is only possible by numerical means, and constitute demanding work. Analytical methods, even approximate, are often advantageous when it is required to follow general trends as temperature and magnetic field vary (for review of previous achievements see Ref. 3). The $H_{c2}(T)$ line can be calculated from the microscopic BCS theory of superconductivity: in some symmetric cases exactly and more often approximately. Below we argue that the eigenvalue of the $H_{c2}(T)$ problem can be used as a small parameter. We consider an illustrative example of a clean metal with spherical Fermi surface (FS) and isotropic pairing interaction.

We start with the equations for the quasiclassical Green functions ($\omega_n + v D/2) f = \Delta g$, $(\omega_n - v D^*/2) \bar{f} = \Delta^*$ complemented by the constraint $f \bar{f} + g^2 = 1$ and the gap equation $\Delta = 2 \pi t \sum_{\omega_n > 0} \left[ \frac{\Delta}{\omega_n} - f_{FS} \bar{f} \right]$. The notations are standard. The unit vector $v$, specified by spherical angles $(\theta, \varphi)$, gives the direction of the Fermi velocity. In this paper we concen-
trate on strongly type II superconductors with the Ginzburg–Landau (GL) parameter $\kappa \gg 1$ which allows to disregard the supercurrent equation. The free energy of a superconductor is taken in the variational form introduced by Eilenberger.\textsuperscript{8} We use the following units: $T_c$ for temperature ($t = T/T_c$ is reduced temperature) and the gap function, $R \equiv \hbar v_F/T_c$ for distances and $H_R \equiv \hbar v_F/e^*R^2$ for magnetic field ($b = B/H_R$ is reduced magnetic induction). The free energy density is given in units of $N_0(\pi T_c)^2$ with $N_0$ being the density of states of a normal metal.

Making use of the smallness of $\Delta$ near $H_{c2}(t)$ line, we proceed according to the following strategy. First, we solve the Eilenberger equations for $f$ and $g$ perturbatively in $\Delta$ and obtain the gap equation solely in terms of $\Delta$. Next, the eigenvalue problem given by the linearized gap equation is worked out. After $H_{c2}(t)$ line is known the expansion parameter $a_h$ can be defined and used to find $\Delta$. In this paper we demonstrate how to obtain the coefficients of the $a_h$ expansion in general and calculate them, along with the free energy, to the lowest order. The Green functions are then also known to the same order.

Step 1. The structure of Eilenberger equations suggests the expansions: $g = g_0 + g_2 + \ldots$, $f = f_1 + f_3 + \ldots$. Subscripts signify the power of $\Delta$ to which the terms are proportional. Using the “inversion” operator $\hat{P} = 1/(\omega_n + \mathbf{v} \cdot \mathbf{D}/2) \equiv \int_0^\infty d\tau \exp \left[-\tau \left(\omega_n + \mathbf{v} \cdot \hat{D}/2\right)\right]$ (note that $\hat{P}'(\mathbf{v}) = \hat{P}^*(-\mathbf{v})$) and starting from the known $g_0 = \text{sgn}(\omega_n)$ we obtain:

\begin{align}
  f_1 &= g_0 \hat{P} \Delta, \quad g_2 = -\frac{g_0}{2} \left(\hat{P} \Delta\right) \left(\hat{P}' \Delta^*\right), \\
  f_3 &= -\frac{g_0}{2} \left(\Delta \left(\hat{P} \Delta\right) \left(\hat{P}' \Delta^*\right)\right), \quad \ldots
\end{align}

Step 2. We rewrite the gap equation as $\mathcal{H} \Delta = a_h \Delta + 2\pi t \sum_{\omega_n > 0} \mathcal{F}_{FS}(f_3 + f_5 + \ldots)$, where $f_3, f_5, \text{etc.}$, have to be taken from Eqs. (1)–(2) and the other quantities are given by

\begin{align}
  a_h &= \ln \frac{1}{t} - 2\pi t \sum_{\omega_n > 0} \left[\frac{1}{\omega_n} - \frac{1}{2} \int_0^\pi d\theta \sin \theta \int_0^\infty d\tau e^{-\tau \omega_n - \tau^2 A} \right], \\
  \mathcal{H} &= -\pi t \sum_{\omega_n > 0} \int_0^\pi d\theta \sin \theta \int_0^\infty d\tau e^{-\tau \omega_n - \tau^2 A} \sum_{m=1}^\infty \frac{(-2\tau A)^m}{(m!)^2} \left(\hat{a}^\dagger\right)^m \hat{a}^m
\end{align}

with $A \equiv b \left(\frac{\sin \theta}{4}\right)^2$. Note that $a_h$ is in fact the eigenvalue of the $H_{c2}$ problem and the $H_{c2}(t)$ line is given by $a_h(t, b) = 0$, which is identical to the original result of Helfand and Werthamer.\textsuperscript{4} For an arbitrary $(t, b)$ point the parameter $a_h$ specifies the “distance” to the $H_{c2}(t)$ line. In the vicinity of