Hilbert Problems (Almost) 100 Years Later
(From the Viewpoint of Interval Computations)

OLGA M. KOSHELEVA
Department of Electrical and Computer Engineering, University of Texas at El Paso, El Paso,
TX 79968, USA, e-mail: olga@ece.utep.edu

Hilbert problems. In 1900, the world-wide mathematical community asked David
Hilbert, the leading mathematician of his time, to prepare the list of challenging
mathematical problems for the coming 20th century. After careful analysis, Hilbert
selected 23 problems that he considered to be the most important and the most
promising. These problems were delivered in his famous 1900 lecture [12].

From the mathematical viewpoint, Hilbert's selection was extremely successful.
The short list of problems indeed inspired breakthroughs and developments that
formed the core of the 20th century mathematics. Some of these problems even
mentioned the notion of what we would now call an algorithm, the notion that was
still in infancy in 1900, but which now shapes the world of science and the humanity
in general.

Hilbert problems: a view from the year (almost) 2000. In the course of these
developments, practically all Hilbert's problems were solved [6]. Solved—in the
sense in which Hilbert himself would like them to be solved: if he asked whether a
certain mathematical object existed, he meant to obtain a proof that such an object
existed, not necessarily an explicit construction. Of course, a construction is always
desirable but back then in 1900, before computers were invented, even the existence
of a construction did not necessarily mean that we could actually construct: it could
be that this construction required thousands of steps, which was, at that time, next
to impossible.

Nowadays, computers routinely make millions and billions of steps, so a the-
oretical ability to construct most frequently means that we can actually construct
(if not now, then in the near future). With this fascinating computational ability
in mind, it is desirable to re-visit year-1900 Hilbert problems from this year-2000
computational viewpoint.

What exactly is the question? Some Hilbert problems are already formulated in
computational terms: e.g., the 10th problem asks to design a process (we would
now call it an algorithm) to check whether a given Diophantine equation has a
solution (Matiyasevich proved [21] that there is no such algorithm). However, most
problems are formulated in non-algorithmic terms. In most cases, even algorithmic
re-formulation is not very straightforward:
• If the question is about the existence of a natural or a rational number, then it is natural to reformulate it as a question about the algorithm for producing such numbers.

• But what if the question is about the existence of a real number? Or a continuous function from real numbers to real numbers? What do we then mean by computing a number or a function?

These questions form the basis of the so-called constructive mathematics (see, e.g., [1–5, 19]) In particular, to compute a real number \( x \) means to be able, for any given accuracy \( k \), to produce an interval of width \( 2^{-k} \) that is guaranteed to contain this number.

In terms of these formalizations, the question is: we know that an object exists; can we have an algorithm for computing this object?

For different Hilbert problems, this question has different answers.

First example: algorithm expected, algorithm found. The first Hilbert problem for which this question was successfully solved was 17th: *To show that if a rational function \( r(x_1, \ldots, x_n) = p(x_1, \ldots, x_n) / q(x_1, \ldots, x_n) \) of several real variables \( x_1, \ldots, x_n \) is always non-negative, then it can be represented as a sum of squares of rational functions: \( r(x_1, \ldots, x_n) = r_1^2(x_1, \ldots, x_n) + \cdots + r_m^2(x_1, \ldots, x_n) \). This problem was solved (by E. Artin) in 1927, but Artin’s proof did not provide any algorithm for finding \( r_1, \ldots, r_m \). The general belief was that such an algorithm must exist, and many researchers tried to find it, but it was actually found only in 1957, by G. Kreisel [15, 16] (see also [8, 9, 20]).

In this problem, researchers expected an algorithm, and an algorithm was actually found.

Second example: no algorithm expected, no algorithm possible. Let us give an example of a problem where an algorithm was not expected: 3rd, about the axiomatization of volume in elementary geometry. Traditional description of a volume requires not only additivity but also the so-called method of exhaustion.

• In the plane, every two polygons of equal area are equi-decomposable, and therefore, any additive function on polygons is either an area or a function of an area.

• In 3D case, in 1900, it was not known whether any two polytopes of equal volume are equi-decomposable.

Dehn has proved that some are not, and moreover, he proved the existence of an additive function that is neither a volume nor a function of a volume. This function was strongly non-constructive (used axiom of choice), and it was widely believed that no constructive function of this type is actually possible. This was proven in [14].

In this problem, researchers expected that no algorithm is possible, and this impossibility was indeed proven.